
R. S. Jain† and M. B. Dhakne‡

†School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded 431606, India
e-mail: rupalisjain@gmail.com
‡Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, India
e-mail: mbdhakne@yahoo.com

Abstract: The aim of the present paper is to establish the global existence of solutions of initial value problem for nonlinear neutral functional integro-differential equations with nonlocal condition in Banach spaces. Our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solution and the Pachpatte’s inequality.

Keywords: integro-differential equation; fixed point; Pachpatte’s inequality.

2010 Mathematics Subject Classification: 45J05; 45N05; 47H10; 47B38.

1 Introduction

The theory of functional differential equations with nonlocal conditions has been extensively studied in the literature, see [1–7] as they have applications in physics and many other areas of applied mathematics. The nonlocal condition is more precise for describing natural phenomena than the classical condition because more information is taken into account, thereby decreasing the negative effects...
incurred by a possibly single measurement taken at initial time. Let $X$ be a Banach space with the norm $\| \cdot \|$. Let $C = C([-r, 0], X), 0 < r < \infty$, be the Banach space of all continuous functions $\psi : [-r, 0] \to X$ endowed with supremum norm $\| \psi \|_C = \sup\{\| \psi(t) \| : -r \leq t \leq 0\}$.

Let $B = C([-r, T], X), T > 0$, be the Banach space of all continuous functions $x : [-r, T] \to X$ with the supremum norm $\| x \|_B = \sup\{\| x(t) \| : -r \leq t \leq T\}$. For any $x \in B$ and $t \in [0, T]$, we denote $x_t$ the element of $C$ given by $x_t(\theta) = x(t + \theta)$, for $\theta \in [-r, 0]$. Consider the nonlinear functional integro-differential equations with nonlocal condition of the type

$$\frac{d}{dt}[x(t) - u(t, x_t)] + Ax(t) = f(t, x_t, \int_0^t k(t,s)h(s,x_s)ds), \quad t \in [0, T],$$

(1.1)

$$x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0,$$

(1.2)

where $-A$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \geq 0$ in $X$, $f : [0, T] \times C \times X \to X$, $k : [0, T] \times [0, T] \to \mathbb{R}$, $h : [0, T] \times C \to X$, $u : [0, T] \times C \to X$ are continuous functions and $\phi$ is given element of $C$. The function $g : C^p \to C$, where $0 < t_1 < t_2 < ... < t_p \leq T$, $p \in \mathbb{N}$ is given function. For example

$$g(x_{t_1}, ..., x_{t_p})(\theta) = \sum_{k=1}^{p} c_k x(t_k + \theta), \quad x \in C([-r, T], X), \quad \theta \in [-r, 0].$$

where $c_k, (k = 1, 2, 3, ..., p)$ are constants.

Equations of the form (1.1)-(1.2) or their special forms serve as an abstract formulation of partial integro-differential equations which arise in the problems with memory visco-elasticity and many other physical phenomena, see [6, 8–10] and the references given therein. The problem of existence, uniqueness and other properties of solutions of the equations of the type (1.1)-(1.2) or their special forms have been studied by many authors by using different techniques, see [1, 5, 7, 11–15].

The aim of the present paper is to study the existence of global solutions to integral equations (1.1)-(1.2). The main tool used in our analysis is based on an application of the topological transversality theorem known as Leray-Schauder alternative, rely on a priori bounds of solutions and Pachpatte’s inequality. The interesting and useful aspect of the method employed here is that it yields simultaneously the global existence of solutions and the maximal interval of existence. This paper is organized as follows. Section 2 presents the preliminaries and the statement of our main result. In Section 3 we prove our main result.

2 Preliminaries and Main Result

Before proceeding to the statement of our main result, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.
Definition 2.1. A function $x : (-r, T) \to X$, $T > 0$ is called the mild solution of the Cauchy problem (1.1)-(1.2) if $x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \ -r \leq t \leq 0$, the restriction of $x(.)$ to the interval $[0, T)$ is continuous and for each $0 \leq t < T$, the function $AT(t-s)u(s,x_s)$, $s \in [0,t]$, is integrable and the integral equation

$$x(t) = T(t)[\phi(0) - g(x_{t_1}, ..., x_{t_p})(0) - u(0, \phi(0) - g(x_{t_1}, ..., x_{t_p})(0))] + u(t,x_t) + \int_0^t AT(t-s)u(s,x_s)ds + \int_0^t T(t-s)f(s,x_s)\int_s^t k(s,\tau)h(\tau,x_\tau)d\tau)ds,$$

$t \in [0, T]$, is satisfied.

The following Pachpatte’s inequality plays the key role in our analysis.

Lemma 2.2. [16] Let $u$, $f$ and $g$ be non-negative continuous functions defined on $\mathbb{R}_+$, for which the inequality

$$u(t) \leq u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)(\int_s^t g(\sigma)u(\sigma)d\sigma)ds, \quad t \in \mathbb{R}_+,$$

holds, where $u_0$ is non-negative constant. Then

$$u(t) \leq u_0[1 + \int_0^t f(s)exp(\int_s^t [f(\sigma) + g(\sigma)]d\sigma)ds], \quad t \in \mathbb{R}_+.$$

Our main result is based on the following version of the topological transversality theorem, known as Leray-Schauder alternative given by Granas [17].

Lemma 2.3. Let $S$ be a convex subset of a normed linear space $E$ and assume $0 \in S$. Let $F : S \to S$ be a completely continuous operator, and let

$$\varepsilon(F) = \{x \in S : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\varepsilon(F)$ is unbounded or $F$ has a fixed point.

We list the following hypotheses for our convenience.

$(H_1)$ $-A$ is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t), t \geq 0$ in $X$. Then there exist constants $K_0 \geq 1$ and $K_1 > 0$ such that

$$\|T(t)\| \leq K_0 \quad \text{and} \quad \|AT(t)\| \leq K_1.$$

$(H_2)$ There exists a continuous function $p : [0, T] \to \mathbb{R}_+ = [0, \infty)$ such that

$$\|f(t, \psi, x)\| \leq p(t)(\|\psi\|_C + \|x\|),$$

for every $t \in [0, T], \psi \in C$ and $x \in X$.

$(H_3)$ There exists a continuous function $q : [0, T] \to \mathbb{R}_+$ such that

$$\|h(t, \psi)\| \leq q(t)(\|\psi\|_C).$$
There exists a constant $G \geq 0$ such that
$$\max \|g(x_{t_1}, x_{t_2}, \ldots, x_{t_p})\| \leq G.$$  \hfill (H_4)

There exist constants $c_1 < 1$ and $c_2 > 0$ such that
$$\|u(t, \phi)\| \leq c_1 \|\phi\| + c_2, \quad t \in [0, T], \quad \phi \in C.$$  \hfill (H_5)

For every positive integer $k$ there exists $h_k \in L^1(0, T)$
$$\sup_{\|\psi\| \leq k} \|f(t, \psi, x)\| \leq h_k(t), \text{ for } t \in [0, T] \text{ a.e.}$$  \hfill (H_6)

For each $t \in [0, T]$ the function $f(t, \psi, x) : [0, T] \to X$ is strongly measurable.

With these preparations we state our main result to be proved in the present paper.

**Theorem 2.4.** Suppose that hypotheses (H_1)-(H_8) are satisfied then the Cauchy problem (1.1)-(1.2) has a mild solution $x$ on $[-r, T]$.

**Remark 2.5.** It is to be noted that J.P. Daur and K. Bhalchandran [11] and M. B. Dhakne and G. B. Lamb [14] have studied equations like (1.1)-(1.2) by using Schaefer’s fixed point theorem and Leray Schauder alternative with local condition respectively. We further remark that here in our theorem, we are achieving results for nonlocal problem with less hypotheses only by applying Pachpatte’s inequality in addition to topological transversality theorem known as Leray Schauder alternative.

### 3 Proof of Theorem

To prove the existence of mild solution of the initial-value problem (1.1)-(1.2), we apply topological transversality theorem given in Theorem 2.4. First we establish the priori bounds on the solutions to the initial value problem (1) for $\lambda \in (0, 1)$, where

$$\frac{d}{dt}[x(t) - \lambda u(t, x_t)] + Ax(t) = \lambda f(t, x_t, \int_0^t k(t, s)h(s, x_s)ds), \quad t \in [0, T]. \quad (1.1)_\lambda$$

Let $x(t)$ be a solution of the problem (1.1)_\lambda-(1.2) then it satisfies the equivalent integral equation

$$x(t) = \lambda T(t)[\phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0) - u(0, \phi(0) - g(x_{t_1}, \ldots, x_{t_p})(0))] + \lambda u(t, x_t)$$

$$+ \lambda \int_0^t \int_0^s A T(t - s)u(s, x_s)ds + \lambda \int_0^t T(t - s)f(s, x_s, \int_0^s k(s, \tau)h(\tau, x_\tau)d\tau)ds,$$

(3.1)
On Global Existence of Solutions for Abstract Nonlinear ...

793

(3.2)

Since $k$ is continuous on compact set $[0, T] \times [0, T]$, there is constant $L > 0$ such that $|k(t, s)| \leq L$, for $0 \leq s \leq t \leq T$. Also let $\|\phi\| = c$. Using hypotheses $(H_1) - (H_5)$ and equation (3.1), we obtain

\[
\|x(t)\| \leq K_0[c + G + c_1(c + G) + c_2] + c_1 \|x_t\|_C + c_2 + \int_0^t K_1[c_1 \|x_s\|_C + c_2]ds
\]

\[
+ \int_0^t K_0p(s)[\|x_s\|_C + \int_0^s |k(s, t)||h(\tau, x_\tau)|d\tau]ds
\]

\[
\leq K_0[(c + G) + c_1(c + G) + c_2] + c_1 \|x_t\|_C + c_2
\]

\[
+ \int_0^t K_1c_1 \|x_s\|_C ds + K_1c_2 \int_0^t ds
\]

\[
+ \int_0^t K_0p(s)[\|x_s\|_C + \int_0^s Lq(\tau)(\|x_\tau\|_C)d\tau]ds
\]

\[
\leq c_3 + c_1 \|x_t\|_C + \int_0^t [K_1c_1 + K_0p(s)]\|x_s\|_C ds
\]

\[
+ \int_0^t K_0p(s) \int_0^s Lq(\tau)(\|x_\tau\|_C)d\tau ds
\]...
where \( Q \) is some constant.

Consequently, we have \( \|x\|_B = \sup\{\|x(t)\| : t \in [-r, T]\} \leq Q \).

Now, we rewrite initial value problem (1.1)-(1.2) as follows: For \( \phi \in C \), define \( \hat{\phi} \in B \) by

\[
\hat{\phi}(t) = \begin{cases} 
\phi(t) - (g(x_{t_1}, ..., x_{t_p}))(t) & \text{if } -r \leq t \leq 0 \\
T(t)[\phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)] & \text{if } 0 \leq t \leq T.
\end{cases}
\]

If \( y \in B \) and \( x(t) = y(t) + \hat{\phi}(t), t \in [-r, T] \), then it is easy to see that \( y \) satisfies

\[
y(t) = y_0 = 0; \quad -r \leq t \leq 0 \quad \text{and}
\]

\[
y(t) = -T(t)[u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))] + u(t, y_t + \hat{\phi}_t)
+ \int_0^t AT(t-s)u(s, y_s + \hat{\phi}_s)ds
+ \int_0^t T(t-s)f(s, y_s + \hat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_{\tau} + \hat{\phi}_\tau)d\tau)ds, \quad t \in [0, T]
\]

if and only if \( x(t) \) satisfies the following integral equations

\[
x(t) = T(t)[\phi(0) - g(x_{t_1}, ..., x_{t_p}))(0)] - u(0, \phi(0) - g(x_{t_1}, ..., x_{t_p}))(0)] + u(t, x_t)
+ \int_0^t AT(t-s)u(s, x_s)ds + \int_0^t T(t-s)f(s, x_s, \int_0^s k(s, \tau)h(\tau, x_{\tau})d\tau)ds,
\]

\( t \in [0, T] \).

\[
x(t) + (g(x_{t_1}, ..., x_{t_p}))(t) = \phi(t), \quad -r \leq t \leq 0.
\]

We define the operator \( F : B_0 \rightarrow B_0, B_0 = \{ y \in B : y_0 = 0 \} \) by

\[
(Fy)(t) = \begin{cases} 
0 & \text{if } -r \leq t \leq 0 \\
-T(t)[u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))] + u(t, y_t + \hat{\phi}_t)
+ \int_0^t AT(t-s)u(s, y_s + \hat{\phi}_s)ds
+ \int_0^t T(t-s)f(s, y_s + \hat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_{\tau} + \hat{\phi}_\tau)d\tau)ds & \text{if } t \in [0, T].
\end{cases}
\]

From the definition of an operator \( F \) defined by the equation (3.7), it is to be noted that the equations (3.5)-(3.6) can be written as

\[
y = Fy
\]

and the integral equations (3.1)-(3.2) can be written as

\[
y = \lambda Fy.
\]
Now, we prove that $F$ is completely continuous. First, we prove that $F : B_0 \to B_0$ is continuous. Let $\{v_n\}$ be a sequence of elements of $B_0$ converging to $v$ in $B_0$. Then there exists an integer $N$ such that $\|v_n(t)\| \leq N$ for all $n$ and $t \in [0, T]$. So $v_n \in B_N$ and $v \in B_N$. Then by using hypotheses $(H_6)$-$(H_8)$ we have

$$f \left( t, v_n + \hat{\phi}_t, \int_0^t k(t, s)h(s, v_n + \hat{\phi}_s)ds \right) \to f \left( t, v + \hat{\phi}_t, \int_0^t k(t, s)h(s, v + \hat{\phi}_s)ds \right)$$

for each $t \in [0, T]$, since

$$\|f \left( t, v_n + \hat{\phi}_t, \int_0^t k(t, s)h(s, v_n + \hat{\phi}_s)ds \right) - f \left( t, v + \hat{\phi}_t, \int_0^t k(t, s)h(s, v + \hat{\phi}_s)ds \right)\|
\leq 2h_{N'}(t),$$

where $N' = \max\{N + \|\hat{\phi}\|, TM'|N + \|\hat{\phi}\|\}$. Then by dominated convergence theorem, we have

$$\|(Fv_n)(t) - (Fv)(t)\| \leq \|u(t, v_n + \hat{\phi}_t) - u(t, v + \hat{\phi}_t)\|
+ \int_0^t \|AT(t - s)\| \|u(s, v_n + \hat{\phi}_s) - u(s, v + \hat{\phi}_s)\|ds
+ \int_0^t \|T(t - s)\| \|f(s, v_n + \hat{\phi}_s, \int_0^t k(s, \tau)h(\tau, v_n + \hat{\phi}_\tau)d\tau) - f(s, v + \hat{\phi}_s, \int_0^t k(s, \tau)h(\tau, v + \hat{\phi}_\tau)d\tau)\|ds
\to 0 \quad \text{as} \quad n \to \infty, \forall t \in [0, T],$$

which implies $Fv_n \to Fv$ in $B_0$ as $v_n \to v$ in $B_0$. Therefore, $F$ is continuous.

To prove that $F$ is completely continuous it remains to prove that $F$ maps a bounded set of $B_0$ into a precompact set of $B_0$. Let $B_m = \{y \in B_0 : \|y\| \leq m\}$ for $m \geq 1$. Now we show that $F_{B_m}$ is uniformly bounded. From the equation $(3.7)$ and using hypotheses $(H_1)$-$(H_5)$ and the fact that $\|y\| \leq m, y \in B_m$ implies $\|y\| \leq m, t \in [0, T]$ we obtain

$$\|(Fy)(t)\| \leq \|T(t)\| \|c_1\phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0)\| + c_2 + c_1\|y + \hat{\phi}\| + c_2
+ \int_0^t K_1[c_1\|y + \hat{\phi}\| + c_2]ds
+ \int_0^t K_0p(s)\|y + \hat{\phi}\|\|c + \int_0^t k(s, \tau)\|h(\tau, y + \hat{\phi}_{\tau})\|d\tau\|ds
\leq K_0[c_1(c + G) + c_2 + c_1(m + c + G) + c_2 + K_1c_1(m + c + G)T
+ K_1c_2T + K_0TM'[m + c + G + \frac{T}{2}(m + c + G)].$$
This implies that the set \( \{(Fy)(t) : \|y\|_B \leq m, \quad -r \leq t \leq T\} \) is uniformly bounded in \( X \) and hence \( F_{B_m} \) is uniformly bounded. Now we show that \( F \) maps \( B_m \) into an equicontinuous family of functions with values in \( X \). Let \( y \in B_m \) and \( t_1, t_2 \in [-r, T] \). Then from the equation (3.7) and using the hypotheses \((H_1)-(H_5)\) we have

Case 1: Suppose \( 0 \leq t_1 \leq t_2 \leq T \)

\[
\|(Fy)(t_2) - (Fy)(t_1)\| \\
\leq \|T(t_1) - T(t_2)\|\|u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))\| \\
+ \|u(t_2, y_{t_2} + \hat{\phi}_{t_2}) - u(t_1, y_{t_1} + \hat{\phi}_{t_1})\| \\
+ \int_0^{t_1} \|AT(t_2 - s) - AT(t_1 - s)\|\|u(s, y_s + \hat{\phi}_s)\|ds \\
+ \int_0^{t_1} \|AT(t_2 - s)\|\|u(s, y_s + \hat{\phi}_s)\|ds \\
+ \int_0^{t_2} \|T(t_2 - s) - T(t_1 - s)\|M^*[m + c + G + TM^*(m + c + G)]ds \\
+ \int_0^{t_1} \|T(t_2 - s)\|M^*[m + c + G + TM^*(m + c + G)]ds.
\]

Case 2: Suppose \(-r \leq t_1 \leq 0 \leq t_2 \leq T\). Then \((Fy)(t_1) = 0\). Therefore proceeding as in Case 1, we get

\[
\|(Fy)(t_2) - (Fy)(t_1)\| = \|(Fy)(t_2)\| \\
\leq \| - T(t_2)[u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))] + u(t_2, y_{t_2} + \hat{\phi}_{t_2})\| \\
+ \int_0^{t_2} \|AT(t_2 - s)u(s, y_s + \hat{\phi}_s)ds\| \\
+ \int_0^{t_2} \|T(t_2 - s)f(s, y_s + \hat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \hat{\phi}_\tau)d\tau)\|ds.
\]

As \( t_2 - t_1 \to 0 \) implies \( t_2 \to t_1 \) but \( t_1 \leq 0 \) and \( t_2 \geq 0 \) implies \( t_2 \to 0 \). Therefore above equation becomes,

\[
\|(Fy)(t_2) - (Fy)(t_1)\| \leq \| - T(0)[u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))] + u(0, y_0 + \hat{\phi}_0)\| \\
\leq \| - [u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))] + u(0, x(0))\| \\
\leq \| - [u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))] + [u(0, \phi(0) - (g(x_{t_1}, ..., x_{t_p}))(0))]\|.
\]

Case 3: Suppose \(-r \leq t_1 \leq t_2 \leq 0\). Then \((Fy)(t_1) = 0\) and \((Fy)(t_2) = 0\). Therefore,

\[
\|(Fy)(t_2) - (Fy)(t_1)\| = 0.
\]

Since \( T(t), \quad t > 0 \) is compact, \( T(t) \) is continuous in uniform operator topology. Therefore, as \( t_2 - t_1 \to 0 \), the right hand side in the cases 1–3 tends to zero. Thus \( F \) maps \( B_m \) into an equicontinuous family of functions with values in \( X \).
We have already shown that $F_{B_m}$ is an equicontinuous and uniformly bounded collection. To prove the set $F_{B_m}$ is precompact in $B$, it is sufficient, by Arzela-Ascoli’s argument, to show that $F$ maps $B_m$ into a precompact set in $X$.

Let $0 < t \leq T$ be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$. For $y \in B_m$, we define

$$(F_{\epsilon}y)(t) = -T(t)[u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0))] + u(t, y_t + \hat{\phi}_t)$$

$$+ \int_0^{t-\epsilon} AT(t-s)u(s, y_s + \hat{\phi}_s)ds$$

$$+ \int_0^{t-\epsilon} T(t-s)f \left( s, y_s + \hat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \hat{\phi}_\tau)d\tau \right) ds$$

$$= -T(t)[u(0, \phi(0) - (g(x_{t_1}, \ldots, x_{t_p}))(0))] + u(t, y_t + \hat{\phi}_t)$$

$$+ T(\epsilon) \int_0^{t-\epsilon} AT(t-s)u(s, y_s + \hat{\phi}_s)ds$$

$$+ T(\epsilon) \int_0^{t-\epsilon} T(t-s-\epsilon) \left( s, y_s + \hat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \hat{\phi}_\tau)d\tau \right) ds.$$ 

Since $T(t)$ is the compact operator, the set $Y_{\epsilon}(t) = \{(F_{\epsilon}y)(t) : y \in B_m\}$ is precompact in $X$ for every $\epsilon$, $0 < \epsilon < t$. Moreover for every $y \in B_m$, we have

$$(Fy)(t) - (F_{\epsilon}y)(t) = \int_0^t AT(t-s)u(s, y_s + \hat{\phi}_s)ds$$

$$+ \int_0^t T(t-s)f \left( s, y_s + \hat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \hat{\phi}_\tau)d\tau \right) ds. $$

By making use of hypotheses $(H_1) - (H_5)$ and the fact that $\|y\|_B \leq m, y \in B_m$ implies $\|y\|_C \leq m, t \in [0, T]$, we have

$$\|(Fy)(t) - (F_{\epsilon}y)(t)\|$$

$$\leq \int_0^t ||AT(t-s)u(s, y_s + \hat{\phi}_s)||ds$$

$$+ \int_0^t ||T(t-s)f \left( s, y_s + \hat{\phi}_s, \int_0^s k(s, \tau)h(\tau, y_\tau + \hat{\phi}_\tau)d\tau \right)||ds$$

$$\leq \int_0^t K_1[c_1(m + c + G) + c_2]ds$$

$$+ \int_0^t ||T(t-s)||p(s) \left[ ||y_s + \hat{\phi}_s||_C + \int_0^s L_q(\|y_\tau + \hat{\phi}_\tau\|_C)d\tau \right]ds$$

$$\leq \{K_1[c_1(m + c + G) + c_2] + K_0M^*[m + c + G + TM^*(m + c + G)]\} \epsilon.$$ 

This shows that there exists precompact sets arbitrarily close to the set $\{(Fy)(t) : y \in B_m\}$. Hence the set $\{(Fy)(t) : y \in B_m\}$ is precompact in $X$. This shows that
$F$ is completely continuous operator. Moreover, the set

$$
\varepsilon(F) = \{ y \in B_0 : y = \lambda F y, \quad 0 < \lambda < 1 \},
$$

is bounded in $B$, since for every $y$ in $\varepsilon(F)$, the function $x(t) = y(t) + \hat{\phi}(t)$ is a solution of initial value problem (1.1)-(1.2) for which we have proved that $\|x\|_B \leq Q$ and hence $\|y\|_B \leq Q + c + G$. Now, by virtue of Lemma 2.3, the operator $F$ has a fixed point $\tilde{y}$ in $B_0$. Then $\tilde{x} = \tilde{y} + \hat{\phi}$ is a solution of the initial value problem (1.1)-(1.2). This completes the proof of the Theorem 2.4.

References


(Received 6 June 2013)
(Accepted 2 June 2016)