The Generalized Stability of an $n$-Dimensional Jensen Type Functional Equation

J. Tipyan, C. Srisawat, P. Udomkavanich and P. Nakmahachalasint

Department of Mathematics, Faculty of Science, Chulalongkorn University
254 Phayathai Road, Pathumwan, Bangkok 10330, Thailand
e-mail: tipyan.j@gmail.com (J. Tipyan)
           Paisan.N@chula.ac.th (P. Nakmahachalasint)

Abstract: In this paper, we will investigate the generalized Hyer-Ulam-Rassias stability of an $n$-dimensional functional equation,

$$\sum_{i=1}^{n} p_i f(x_i) = f\left(\sum_{i=1}^{n} p_i x_i\right),$$

where $n > 1$ is an integer, and $p_1, \ldots, p_n$ are positive real number with

$$\sum_{i=1}^{n} p_i = 1.$$

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1 Introduction and preliminaries

In 1940 S.M. Ulam proposed the famous stability problem of linear functions. In 1941 D.H. Hyers considered the case of an approximately additive function $f : E \to E'$ where $E$ and $E'$ are Banach spaces and $f$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

Corresponding author.

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for all $x, y \in E$ and for some $\varepsilon > 0$. It was shown that the limit

$$L(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive function satisfying

$$\|f(x) - L(x)\| \leq \varepsilon.$$ 


$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$ where $\theta \geq 0$ and $0 \leq p < 1$ are constants. Since then, the stability problem has been widely investigated for different types of functional equations. The Jensen functional equation given by

$$\frac{f(x) + f(y)}{2} = f\left(\frac{x + y}{2}\right)$$

has close connection [3, 4] with the Cauchy functional equation

$$f(x) + f(y) = f(x + y).$$

Stability of Jensen equation has been studied at first by Kominek [7]. In 1998, S.M. Jung [5] investigated the Hyers-Ulam stability for Jensen’s equation on a restricted domain.

In this paper, we will extend the Jensen functional equation to an $n$-dimensional version,

$$\sum_{i=1}^{n} p_i f(x_i) = f\left(\sum_{i=1}^{n} p_i x_i \right),$$

(1.1)

where $n > 1$ is an integer, and $p_1, \ldots, p_n$ are positive rational numbers with

$$\sum_{i=1}^{n} p_i = 1,$$ 

(1.2)

will study a general solution and investigate its generalized stability. We will also discuss Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

For our convenience, we let $n > 1$ be an integer, $p_1, \ldots, p_n$ be positive rational numbers with (1.2). $X$ be a real vector space and $Y$ be a Banach spaces.

2 Main Results

In this section, we will study the general solution and generalized stability of (1.1). The results are as follows.
2.1 General Solution

**Theorem 2.1.** Let \( X \) and \( Y \) be real vector spaces. A mapping \( f : X \to Y \) satisfies the functional equation (1.1) where \( n > 1 \) is an integer, and \( p_1, \ldots, p_n \) are positive rational numbers with \( \sum_{i=1}^{n} p_i = 1 \) for all \( x_1, \ldots, x_n \in X \), if and only if \( f(x) = A(x) - f(0) \) for all \( x \in X \) where \( A : X \to Y \) is additive function and \( f(0) \) is a constant.

**Proof.** (Necessity) Suppose \( f : X \to Y \) satisfies the functional equation (1.1). Define a function \( g : X \to Y \) by

\[
g(x) = f(x) - f(0)
\]

for all \( x \in X \). Note that \( g(0) = 0 \).

Consider

\[
g\left( \sum_{i=1}^{n} p_i x_i \right) = f\left( \sum_{i=1}^{n} p_i x_i \right) - f(0) = \sum_{i=1}^{n} p_i f(x_i) - f(0)
\]

\[
= \sum_{i=1}^{n} p_i g(x_i).
\]

Thus \( g \) is satisfies (1.1). Let \( s \in \{1, \ldots, n\} \). Then set \( x_s = x \) and \( x_1 = \ldots = x_{s-1} = x_{s+1} = \ldots = x_n = 0 \), then (2.1) becomes

\[
g(p_s x) = p_s g(x) \quad \text{for all} \quad s \in \{1, \ldots, n\} \quad \text{for all} \quad x \in X.
\]

Next, we put \( x_s = x, x_{s+1} = y \) and \( x_1 = \ldots = x_{s-1} = x_{s+2} = \ldots = x_n = 0 \) in (2.1) and using (2.2), we will have

\[
g(p_s x + p_{s+1} y) = p_s g(x) + p_{s+1} g(y)
\]

for all \( x, y \in X \). Therefore \( g \) is additive function, by definition of \( g \) we get \( f(x) = A(x) - f(0) \) for all \( x \in X \).

(Sufficiency) Suppose \( f(x) = A(x) - f(0) \) for all \( x \in X \) where \( A : X \to Y \) is additive function and \( f(0) \) is a constant. Then,

\[
f\left( \sum_{i=1}^{n} p_i x_i \right) = A\left( \sum_{i=1}^{n} p_i x_i \right) - f(0) = \sum_{i=1}^{n} p_i A(x_i) - \sum_{i=1}^{n} p_i f(0)
\]

\[
= \sum_{i=1}^{n} p_i (A(x_i) - f(0)) = \sum_{i=1}^{n} p_i f(x_i).
\]

This completes the proof. \( \Box \)
2.2 Generalized Stability

**Theorem 2.2.** Let \( \phi : X^n \to [0, \infty) \) be a function. For each integer \( s = 1, \ldots, n \), let \( \phi_s : X \to [0, \infty) \) be a function such that

\[
\phi_s(x) = \phi(0, \ldots, 0, x, 0, \ldots, 0)
\]

(2.3)

and \( \sum_{i=0}^{\infty} p_s^{-i} \phi(p_i x) \) converges and \( \lim_{m \to \infty} p_s^{-m} \phi(p_s^m x_1, \ldots, p_s^m x_n) = 0 \) for all \( x_1, \ldots, x_n \in X \).

If a function \( f : X \to Y \) satisfies the inequality

\[
\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \phi(x_1, \ldots, x_n)
\]

(2.4)

for all \( x_1, \ldots, x_n \in X \), then there exists a unique function \( L : X \to Y \) that satisfies functional equation (1.1) and the inequality

\[
\|f(x) - L(x)\| \leq \sum_{i=0}^{\infty} p_s^{-i-1} \phi_s(p_i x)
\]

(2.5)

for all \( x \in X \). The function \( L \) is given by

\[
L(x) = f(0) + \lim_{m \to \infty} p_s^{-m} (f(p_s^m x) - f(0))
\]

(2.6)

for all \( x \in X \).

**Proof.** Suppose \( f : X \to Y \) satisfies the inequality (2.4). Define a function \( g : X \to Y \) by

\[
g(x) = f(x) - f(0)
\]

(2.7)

for all \( x_1, \ldots, x_n \in X \). It should be noted that \( g(0) = 0 \). By (1.2), we get

\[
\sum_{i=1}^{n} p_i g(x_i) - g\left(\sum_{i=1}^{n} p_i x_i\right) \leq \phi(x_1, \ldots, x_n)
\]

(2.8)

for all \( x_1, \ldots, x_n \in X \). Let \( s \in \{1, \ldots, n\} \). Set \( x_s = x \) and \( x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0 \), then (2.8) becomes

\[
\|p_s g(x) - g(p_s x)\| \leq \phi_s(x)
\]

(2.9)

for all \( x \in X \). Rewrite the above equation to

\[
\|g(x) - p_s^{-1} g(p_s x)\| \leq p_s^{-1} \phi_s(x)
\]

(2.10)
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for all \( x \in X \). For each positive integer \( m \) and each \( x \in X \), we have

\[
\| g(x) - p_s^{-m} g(p_s^m x) \| = \left\| \sum_{i=0}^{m-1} \left( p_s^{-i} g(p_s^i x) - p_s^{-(i+1)} g(p_s^{i+1} x) \right) \right\|
\leq \sum_{i=0}^{m-1} \| p_s^{-i} g(p_s^i x) - p_s^{-(i+1)} g(p_s^{i+1} x) \|
= \sum_{i=0}^{m-1} p_s^{-i} \| g(p_s^i x) - p_s^{-1} g(p_s p_s^i x) \|
\leq \sum_{i=0}^{m-1} p_s^{-i-1} \phi_s(p_s^i x). \tag{2.11}
\]

Consider the sequence \( \{p_s^{-m} g(p_s^m x)\} \). For each positive integers \( k < l \) and each \( x \in X \),

\[
\| p_s^{-k} g(p_s^k x) - p_s^{-l} g(p_s^l x) \| = p_s^{-k} \| g(p_s^k x) - p_s^{-(l-k)} g(p_s^{l-k} p_s^k x) \|
\leq p_s^{-k} \sum_{i=0}^{l-k-1} p_s^{-i} \phi_s(p_s^{i+k} x)
\leq p_s^{-k-1} \sum_{i=0}^{\infty} p_s^{-i} \phi_s(p_s^{i+k} x).
\]

Since \( \sum_{i=0}^{\infty} p_s^{-i} \phi_s(p_s^i x) \) converges, \( \lim_{k \to \infty} p_s^{-k-1} \sum_{i=0}^{\infty} p_s^{-i} \phi_s(p_s^{i+k} x) = 0 \); therefore,

\[
L(x) = f(0) + \lim_{m \to \infty} p_s^{-m} g(p_s^m x) \tag{2.12}
\]

is well-defined in the Banach space \( Y \). Moreover, as \( m \to \infty \), (2.11) becomes

\[
\| g(x) + f(0) - L(x) \| \leq \sum_{i=0}^{\infty} p_s^{-i-1} \phi_s(p_s^i x).
\]

Recalling the definition of \( g(x) \), we see that inequality (2.5) is valid.

To show that \( L \) indeed satisfies (1.1), replace each \( x_i \) in (2.8) with \( p_s^m x_i \),

\[
\left\| \sum_{i=1}^{n} p_i g(p_s^m x_i) - g \left( \sum_{i=1}^{n} p_i x_i \right) \right\| \leq \phi(p_s^m x_1, \ldots, p_s^m x_n). \tag{2.13}
\]

If we multiply the above inequality by \( p_s^{-m} \) and take the limit as \( m \to \infty \), then by the definition of \( L \) in (2.12) and (1.2), we obtain

\[
\left\| \sum_{i=1}^{n} p_i L(x_i) - L \left( \sum_{i=1}^{n} p_i x_i \right) \right\| \leq \lim_{m \to \infty} p_s^{-m} \phi(p_s^m x_1, \ldots, p_s^m x_n) = 0, \tag{2.14}
\]
which implies that
\[
\sum_{i=1}^{n} p_i L(x_i) = L \left( \sum_{i=1}^{n} p_i x_i \right) \tag{2.15}
\]
for all \(x_1, \ldots, x_n \in X\).

To prove the uniqueness, suppose there is another function \(L' : X \to Y\) satisfying (1.1) and (2.5). Observe that if we replace \(x_s\) by \(x\) and put \(x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0\) in (2.15), then
\[
p_s L(x) + (1 - p_s) L(0) = L(p_s x) \tag{2.16}
\]
for all \(x \in X\), and
\[
L(0) = f(0) + \lim_{m \to \infty} p_s^{-m} g(0) = f(0).
\]
The function \(L'\) obviously possesses the same properties. Therefore,
\[
p_s (L(x) - L'(x)) = L(p_s x) - L'(p_s x) \tag{2.17}
\]
for all \(x \in X\). We can prove by mathematical induction that for each positive integer \(m\),
\[
p_s^m (L(x) - L'(x)) = L(p_s^m x) - L'(p_s^m x)
\]
for all \(x \in X\). Therefore, for each positive integer \(m\),
\[
\|L(x) - L'(x)\| = p_s^{-m} \|L(p_s^m x) - L'(p_s^m x)\|
\leq p_s^{-m} (\|L(p_s^m x) - f(p_s^m x)\| + \|L'(p_s^m x) - f(p_s^m x)\|)
\leq 2p_s^{-m} \sum_{i=0}^{\infty} p_s^{-i} \phi_s(p_s^{i+m} x)
\]
for all \(x \in X\). Since \(\sum_{i=0}^{\infty} p_s^{-i} \phi(p_s^i x)\) converges, \(\lim_{m \to \infty} p_s^{-m} \sum_{i=0}^{\infty} p_s^{-i} \phi(p_s^{i+m} x) = 0\).

We conclude that \(L(x) = L'(x)\) for all \(x \in X\). \(\square\)

**Theorem 2.3.** Let \(\phi : X^n \to [0, \infty)\) be a function. For each integer \(s = 1, \ldots, n\), let \(\phi_s : X \to [0, \infty)\) be a function such that (2.3) and \(\sum_{i=0}^{\infty} p_s^{-i} \phi(p_s^{-i} x)\) converges and \(\lim_{m \to \infty} p_s^m \phi(p_s^{-m} x_1, \ldots, p_s^{-m} x_n) = 0\) for all \(x_1, \ldots, x_n \in X\). If a function \(f : X \to Y\) satisfies the inequality (2.4) then there exists a unique function \(L : X \to Y\) that satisfies functional equation (1.1) and the inequality
\[
\|f(x) - L(x)\| \leq \sum_{i=1}^{\infty} p_s^{-i} \phi_s(p_s^{-i} x) \tag{2.18}
\]
for all \( x \in X \). The function \( L \) is given by

\[
L(x) = f(0) + \lim_{m \to \infty} p_s^m f(p_s^{-m} x)
\]  
(2.19)

for all \( x \in X \).

**Proof.** Let \( f : X \to Y \) satisfy the inequality \( (2.4) \). Referring the process \( (2.7)-(2.10) \), we can replace inequality \( (2.10) \) with

\[
\| g(x) - p_s g(p_s^{-1} x) \| \leq \phi_s(p_s^{-1} x)
\]

for all \( x \in X \). For each positive integer \( m \) and each \( x \in X \), we get

\[
\| g(x) - p_s^m g(p_s^{-m} x) \| = \left( \sum_{i=1}^{m} p_s^{i-1} g(p_s^{-(i-1)} x) - p_s^i g(p_s^{-i} x) \right)
\leq \sum_{i=1}^{m} \| p_s^{i-1} g(p_s^{-(i-1)} x) - p_s^i g(p_s^{-i} x) \|
= \sum_{i=1}^{m} \| p_s^{i-1} \| g(p_s^{-(i-1)} x) - p_s^i g(p_s^{-i} x) \|
\leq \sum_{i=1}^{m} p_s^{i-1} \phi_s(p_s^{-i} x).
\]  
(2.20)

We investigate the sequence \( \{ p_s^m g(p_s^{-m} x) \} \). For each positive integer \( k < l \) and each \( x \in X \),

\[
\| p_s^k g(p_s^{-k} x) - p_s^l g(p_s^{-l} x) \| = p_s^k \| g(p_s^{-k} x) - p_s^l g(p_s^{-l} k p_s^{-k} x) \|
\leq p_s^k \sum_{i=1}^{l-k} p_s^{i-1} \phi_s(p_s^{-i-k} x)
\leq p_s^{l-1} \sum_{i=0}^{\infty} p_s^i \phi_s(p_s^{-i-k} x).
\]

Since \( \sum_{i=0}^{\infty} p_s^i \phi_s(p_s^{-i} x) \) converges, \( \lim_{k \to \infty} p_s^{k-1} \sum_{i=0}^{\infty} p_s^i \phi_s(p_s^{-i-k} x) = 0 \). Thus,

\[
L(x) = f(0) + \lim_{m \to \infty} p_s^m g(p_s^{-m} x)
\]  
(2.21)

is well-defined in the Banach space \( Y \). Furthermore, \( (2.20) \) becomes as \( m \to \infty \),

\[
\| g(x) + f(0) - L(x) \| \leq \sum_{i=1}^{\infty} p_s^{i-1} \phi_s(p_s^{-i} x).
\]

By the definition of \( g(x) \), inequality \( (2.18) \) is valid.
In order to show that $L$ satisfies (1.1), we replace each $x_i$ by $p_s^{-m}x_i$ in (2.15) and multiply $p_s^m$, then take the limit as $m \to \infty$, we have
\[
\left\| \sum_{i=1}^{n} p_i L(x_i) - L \left( \sum_{i=1}^{n} p_i x_i \right) \right\| \leq \lim_{m \to \infty} p_s^m \phi(p_s^{-m}x_1, \ldots, p_s^{-m}x_n) = 0,
\]
which implies (2.15).

To prove the uniqueness, suppose there is another function $L' : X \to Y$ satisfying (1.1) and (2.5). Replacing $x$ by $p_s^{-1}x$ and put $x_1 = \cdots = x_{s-1} = x_{s+1} = \cdots = x_n = 0$ in (2.15); consequently, (2.17) becomes
\[
p_s(L(p_s^{-1}x) - L'(p_s^{-1}x)) = L(x) - L'(x).\]
For each positive $m$, we can show by mathematical induction that
\[
p_s^m (L(p_s^{-m}x) - L'(p_s^{-m}x)) = L(x) - L'(x)
\]
for all $x \in X$. Therefore, for each positive integer $m$,
\[
\left\| L(x) - L'(x) \right\| = p_s^m \left\| L(p_s^{-m}x) - L'(p_s^{-m}x) \right\|
\leq p_s^m \left( \left\| L(p_s^{-m}x) - f(p_s^{-m}x) \right\| + \left\| L'(p_s^{-m}x) - f(p_s^{-m}x) \right\| \right)
\leq 2p_s^m \sum_{i=1}^{\infty} p_s^{-i-m} \phi(p_s^{-i}x)
\]
for all $x \in X$. Since $\sum_{i=1}^{\infty} p_s^i \phi(p_s^{-i}x)$ converges, $\lim_{m \to \infty} p_s^m \sum_{i=0}^{\infty} p_s^{-i-m} \phi(p_s^{-i-m}x) = 0$. We obtain that $L(x) = L'(x)$ for all $x \in X$. \qed

2.3 Stability

This section will give the stability of (1.1) in various case. The following theorem proves stability of (1.1).

\textbf{Theorem 2.4.} Let $\varepsilon > 0$ be a real number. If a function $f : X \to Y$ satisfies the inequality
\[
\left\| \sum_{i=1}^{n} p_i f(x_i) - f \left( \sum_{i=1}^{n} p_i x_i \right) \right\| \leq \varepsilon
\]
for all $x_1, \ldots, x_n \in X$, then there exists a unique function $L : X \to Y$ that satisfies (1.1) and
\[
\left\| f(x) - L(x) \right\| \leq \frac{\varepsilon}{1 - p_{\text{min}}}
\]
for all $x \in X$, where $p_{\text{min}} = \min\{p_1, \ldots, p_n\}$. 

Proof. Let
\[ \phi(x_1, \ldots, x_n) = \varepsilon \]
for all \( x_1, \ldots, x_n \in X \) in Theorem 2.3. We can see that Theorem 2.3 holds for every \( s = 1, \ldots, n \). We choose \( s \) such that \( p_s = p_{\text{min}} = \min\{p_1, \ldots, p_n\} \). Then (2.18) becomes
\[ \|f(x) - L(x)\| \leq \varepsilon \sum_{i=1}^{n} p_s^{i-1} = \frac{\varepsilon}{1 - p_s} = \frac{\varepsilon}{1 - p_{\text{min}}} \]
for all \( x \in X \) as desired. \( \square \)

The following theorem proves the stability of (1.1).

**Theorem 2.5.** Let \( \varepsilon > 0 \) and \( r > 0 \) be real numbers with \( r \neq 1 \). If a function \( f : X \to Y \) satisfies the inequality
\[ \left\| \sum_{i=1}^{n} p_i f(x_i) - f \left( \sum_{i=1}^{n} p_i x_i \right) \right\| \leq \varepsilon \sum_{i=1}^{n} \|x_i\|^r \]
for all \( x_1, \ldots, x_n \in X \), then there exists a unique function \( L : X \to Y \) that satisfies (1.1) and
\[ \|f(x) - L(x)\| \leq \frac{\varepsilon}{M} \|x\|^r \]
for all \( x \in X \), where \( M = \max_{i=1,\ldots,n} |p_i - p_i^r| \).

Proof. In the case \( 0 < r < 1 \), let
\[ \phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^r \]
for all \( x_1, \ldots, x_n \in X \) in Theorem 2.3. Then we can see that Theorem 2.3 holds for every \( s = 1, \ldots, n \). We choose \( s \) such that
\[ |p_s - p_s^r| = M = \max_{i=1,\ldots,n} |p_i - p_i^r| . \]
Thus, (2.5) becomes
\[
\|f(x) - L(x)\| \leq \varepsilon \sum_{i=1}^{n} p_s^{i-1} \|x_i\|^r \leq \varepsilon \sum_{i=1}^{n} p_s^{i(1-r)} \sum_{i=1}^{n} p_s^{(1-r)} \|x_i\|^r \leq \varepsilon \max_{i=1,\ldots,n} |p_i - p_i^r| \|x\|^r \leq \varepsilon \frac{1}{M} \|x\|^r \]
for all \( x \in X \). In the case \( r > 1 \), let
\[ \phi(x_1, \ldots, x_n) = \varepsilon \sum_{i=1}^{n} \|x_i\|^r \]
for all $x_1, \ldots, x_n \in X$ in Theorem 2.1. Since Theorem 2.1 holds for every $s = 1, \ldots, n$, (2.5) becomes

$$
\|f(x) - L(x)\| \leq \varepsilon \sum_{i=0}^{\infty} p_s^{-i-1} \|p_s^i x\|^r = \varepsilon \|x\|^r \sum_{i=0}^{\infty} p_s^{i(r-1)}
$$

$$
= \varepsilon \|x\|^r p_s^{-1}\left(\frac{1}{1 - p_s^r}\right) = \frac{\varepsilon}{p_s^r - p_s} \|x\|^r = \frac{\varepsilon}{M} \|x\|^r
$$

for all $x \in X$. This completes the proof. \qed

References


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