E-Torsion Free Acts Over Monoids

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Abstract: In this paper we introduce E-torsion freeness of acts over monoids, and will give a characterization of monoids by this property of their (cyclic, monocylic, Rees factor) acts.

Keywords: S-act; E-torsion freeness; flatness.
2010 Mathematics Subject Classification: 20M30.

1 Introduction

Throughout this paper S will denote a monoid. We refer the reader to [1] and [2] for basic definitions and terminology relating to semigroups and acts over monoids and to [3], [4], [5] and [6] for definitions and results on flatness which are used here. A monoid S is called left(right) collapsible if for any s, s′ ∈ S there exists z ∈ S such that zs = z′s (zs = s′z). A submonoid P of S is called weakly left collapsible if for any s, s′ ∈ P, z ∈ S, sz = s′z implies the existence of u ∈ P such that us = us′. It is obvious that every left collapsible submonoid is weakly left collapsible, but the converse is not true. A monoid S is called right (left) reversible, if for any s, s′ ∈ S, there exist u, v ∈ S such that us = vs′ (su = s′v). A submonoid P of S is called weakly right reversible, if for any s, s′ ∈ P, z ∈ S, sz = s′z implies the existence of u, v ∈ P such that us = vs′. A right ideal K_S of a monoid S is called left stabilizing, if for any k ∈ K_S, there exists l ∈ K_S such that lk = k. K_S is called left annihilating, if for any t ∈ S, x, y ∈ S \ K_S, xt, yt ∈ K_S

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implies that $xt = yt$. $K_S$ is called \textit{strongly left annihilating}, if for all $s,t \in S \setminus K_S$ and for all homomorphisms $f : S \to S$, $f(s), f(t) \in K_S$ implies that $f(s) = f(t)$. $K_S$ is called \textit{completely left annihilating}, if for all $x, y, z, t, t' \in S$,

$$[(xt \neq yt') \land (tz = t'z)] \Rightarrow [(xt \notin K_S) \lor (yt' \notin K_S) \lor (x \in K_S) \lor (y \in K_S)]$$

$K_S$ is called $P_{E}$-left annihilating, if for all $x, y, t, t' \in S$,

$$(xt \neq yt') \Rightarrow [(x \in K_S) \lor (y \in K_S) \lor (xt \notin K_S) \lor (yt' \notin K_S) \lor$

$$(\exists u, v \in S, e, f \in E(S), et = t, ft' = t', ut = vt')$

$$xe \neq ue \Rightarrow xe, ue \in K_S, yf \neq vf \Rightarrow yf, vf \in K_S]$$

$K_S$ is called $E$-left annihilating, if for all $x, y, t \in S$,

$$(xt \neq yt) \Rightarrow [(x \in K_S) \lor (y \in K_S) \lor (xt \notin K_S) \lor (yt \notin K_S) \lor$

$$(\exists u, v \in S, e, f \in E(S), et = t, ft = t, ut = vt,$$

$$xe \neq ue \Rightarrow xe, ue \in K_S, yf \neq vf \Rightarrow yf, vf \in K_S]$$

A right $S$-act $A$ satisfies Condition $(P)$, if for all $a, a' \in A, s, s' \in S$, $as = a's'$ implies that there exist $b \in A, u, v \in S$ such that $a = bu$, $a' = bv$ and $us = vs'$. A monoid $S$ is called \textit{right PCP}, if all principal right ideals of $S$ satisfy Condition $(P)$. A right $S$-act $A$ satisfies Condition $(P')$, if for all $a, a' \in A, s, s', z \in S$, $as = a's'$, $sz = s'z$ imply that there exist $b \in A, u, v \in S$ such that $a = bu$, $a' = bv$ and $us = vs'$. A right $S$-act $A$ satisfies Condition $(P_E)$, if for all $a, a' \in A, s, s' \in S$, $as = a's'$ implies that there exist $b \in A, u, v, e^2 = e, f^2 = f \in S$ such that $ae = bue, a'f = bvf, es = s, fs' = s'$ and $us = vs'$. It is obvious that Condition $(P)$ implies Condition $(P_E)$, but not the converse. $A$ satisfies Condition $(E)$, if for all $a \in A, s, s' \in S$, $as = as'$ implies that there exist $b \in A, u \in S$ such that $a = bu$ and $us = us'$. $A$ satisfies Condition $(E')$, if for all $a \in A, s, s', z \in S$, $as = as'$ and $sz = s'z$ implies that there exist $b \in A, u, v \in S$ such that $a = bu$ and $us = vs'$. $A$ satisfies Condition $(E')$, if for all $a \in A, s, s', z \in S$, $as = as'$ implies that there exist $b \in A, u, v \in S$ such that $a = bu$ and $us = vs'$. $A$ satisfies Condition $(E')$, if for all $a \in A, s, s', z \in S$, $as = as'$ and $sz = s'z$ imply that there exist $b \in A, u, v \in S$ such that $a = bu$ and $us = vs'$. It is obvious that $A$ satisfies Condition $(E) \Rightarrow$ Condition $(E') \Rightarrow$ Condition $(E')$. In \cite{2} and \cite{3} we gave a characterization of monoids by Conditions $(E)$ and $(E')$ of their acts. A right $S$-act $A$ satisfies Condition $(PWP)$, if for all $a, a' \in A, s \in S$, $as = a's'$ implies that there exist $b \in A$ and $u, v \in S$ such that $a = bu, a' = bv$ and $us = vs$. A right $S$-act $A$ satisfies Condition $(PWP_E)$, if for all $a, a' \in A, s \in S$, $as = a's'$ implies that there exist $b \in A$ and $u, v, e^2 = e, f^2 = f \in S$ such that $ae = bue, a'f = bvf, es = fs = s$ and $us = vs$. In \cite{4} we gave a characterization of monoids by Conditions $(PWP_E)$ of their acts. A right $S$-act $A$ satisfies Condition $(W)$, if $as = a't$, for $a, a' \in A_S$, $s, t \in S$, implies that there exist $b \in A_S$ and $u \in Ss \cap St$, such that $as = a't = bu$. 

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$A_S$ is called regular, if all cyclic subacts of $A$ are projective. $A_S$ is called faithful, if for $s, t \in S$ the equality $as = at$ for all $a \in A$ implies $s = t$. $A_S$ is called strongly faithful, if for $s, t \in S$ the equality $as = at$ for some $a \in A$ implies $s = t$. $A_S$ is called $P$-regular, if all cyclic subacts of $A$ satisfy Condition ($P$). In [10] we gave a characterization of monoids by $P$-regularity of their acts. $A$ is called strongly $(P)$-cyclic if for any $a \in A$ there exists $z \in S$ such that $\ker \lambda_a = \ker \lambda_z$ and $zS$ satisfies Condition ($P$). In [11] we gave a characterization of monoids $S$ by strong $(P)$-cyclic of right $S$-acts. $A_S$ is called locally cyclic, if every finitely generated subact of $A$ is contained within a cyclic subact of $A$. An act $A_S$ is called to be connected, if for all $a, a' \in A$ there exist elements $s_1, t_1, \ldots, s_n, t_n \in S$ and $a_2, \ldots, a_n \in A$ such that

\[\begin{align*}
as_1 &= a_2 t_1 \\
a_2 s_2 &= a_3 t_2 \\
& \quad \vdots \\
a_n s_n &= a' t_n.\end{align*}\]

For torsionless of acts we refer the reader to [12].

## 2 General Properties

An element $s \in S$ acts injectively on $A_S$ if $as = bs$, for $a, b \in A_S$, implies $a = b$. If every $s \in S$ acts injectively on $A_S$, then we say that $S$ acts injectively on $A_S$.

**Definition 2.1.** An act $A_S$ is called $E$-torsion free (ETF), if $E(S)$ acts injectively on $A_S$, that is;

\[(\forall a, a' \in A_S)(\forall e \in E(S))(ae = a'e \Rightarrow a = a').\]

**Proposition 2.2.** Let $S$ be a monoid. Then:

1. The one-element act $\Theta_S$ is ETF.
2. If $E(S) = \{1\}$, then all (left) right $S$-acts are ETF.
3. $S_S$ is ETF if and only if $E(S) = \{1\}$.
4. If $S$ is a regular monoid, then $A_S$ is ETF if and only if $S$ acts injectively on $A_S$.
5. If $A_i, i \in I$, are right $S$-acts, then $A_i, i \in I$, are ETF if and only if $A_S = \coprod_{i \in I} A_i$ is ETF.
6. If $A_i, i \in I$, are right $S$-acts, then $A_i, i \in I$, are ETF if and only if $A_S = \prod_{i \in I} A_i$ is ETF.
(7) If an act is ETF, then all its subacts are ETF.
(8) $A_S$ is an ETF right $S$-act if and only if $ae = a$, for all $a \in A_S$ and $e \in E(S)$.
(9) If $S = T^1$, where $T$ is a semigroup, then the right $S$-act $T_S$ is ETF if and only if $E(T) = \emptyset$ or $te = t$, for all $t \in T$ and $e \in E(T)$.
(10) If $S$ is an idempotent monoid, then the right $S$-act $A_S$ is ETF if and only if $A_S$ is a coproduct of one element acts.
(11) If $S$ contains a left zero, then the right $S$-act $A_S$ is ETF if and only if $A_S$ is a coproduct of one element acts.

Proof. The statements (1) to (8) are clear from definition.
(9). It follows from (8).
(10). It follows from (5) and (8).
(11). Necessity. Let $z$ be a left zero element of $S$. By (8), $az = a$, for all $a \in A_S$. Thus $as = (az)s = a(zs) = az = a$, for all $s \in S$ and $a \in A_S$. Hence $A_S$ is a coproduct of one element acts.
Sufficiency. It follows from (1) and (5).

3 Characterization by $E$-Torsion Freeness of Right Acts

In this section we characterize monoids by $E$-torsion freeness of right acts.

**Theorem 3.1.** Let $S$ be a monoid and $(U)$ be a property of $S$-acts which $S_S$ has property $(U)$. Then the following statements are equivalent:

1. All right $S$-acts with property $(U)$ are ETF.
2. All finitely generated right $S$-acts with property $(U)$ are ETF.
3. All cyclic right $S$-acts with property $(U)$ are ETF.
4. $E(S) = \{1\}$.

**Proof.** Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4). Since $S_S$ is a cyclic right $S$-act, by assumption it is ETF, and so by Proposition 2.2(3), $E(S) = \{1\}$.
(4) $\Rightarrow$ (1). It follows from Proposition 2.2(2).

Now we have the following corollary.

**Corollary 3.2.** Let $S$ be a monoid. Then the following statements are equivalent:

1. All right $S$-acts are ETF.
2. All torsion free right $S$-acts are ETF.
3. All principally weakly flat right $S$-acts are ETF.
(4) All $GP$-flat right $S$-acts are ETF.
(5) All weakly flat right $S$-acts are ETF.
(6) All right $S$-acts satisfying Condition $(W)$ are ETF.
(7) All flat right $S$-acts are ETF.
(8) All right $S$-acts satisfying Condition $(WP)$ are ETF.
(9) All right $S$-acts satisfying Condition $(PWP)$ are ETF.
(10) All translation kernel flat right $S$-acts are ETF.
(11) All principally weakly kernel flat right $S$-acts are ETF.
(12) All weakly kernel flat right $S$-acts are ETF.
(13) All right $S$-acts satisfying Condition $(P)$ are ETF.
(14) All right $S$-acts satisfying Condition $(PE)$ are ETF.
(15) All right $S$-acts satisfying Condition $(P')$ are ETF.
(16) All right $S$-acts satisfying Condition $(PWP_E)$ are ETF.
(17) All equalizer flat right $S$-acts are ETF.
(18) All strongly flat right $S$-acts are ETF.
(19) All weakly pullback flat right $S$-acts are ETF.
(20) All projective right $S$-acts are ETF.
(21) All projective generators right $S$-acts are ETF.
(22) All generators right $S$-acts are ETF.
(23) All free right $S$-acts are ETF.
(24) All right $S$-acts satisfying Condition $(E)$ are ETF.
(25) All right $S$-acts satisfying Condition $(EP)$ are ETF.
(26) All right $S$-acts satisfying Condition $(E')$ are ETF.
(27) All right $S$-acts satisfying Condition $(E'P)$ are ETF.
(28) All faithful right $S$-acts are ETF.
(29) All torsionless right $S$-acts are ETF.
(30) $E(S) = \{1\}$.

Notice that all statements in Corollary above, are also true for cyclic and finitely generated right $S$-acts.

**Lemma 3.3.** Let $S$ be a monoid. Then the following statements are equivalent:

1. $S$ is left cancellative.
There exists a strongly faithful right $S$-act.

Proof. (1) $\Rightarrow$ (2). It is obvious, because in this case $S_S$ is a strongly faithful right $S$-act.

(2) $\Rightarrow$ (1). Suppose that $A_S$ is a strongly faithful right $S$-act and let $us = ut$, for $u, s, t \in S$. Let $a \in A$. Then $(au)s = (au)t$, and so $s = t$. Thus $S$ is left cancellative, as required.

**Theorem 3.4.** Let $S$ be a monoid and suppose there exists a strongly faithful right $S$-act. Then the following statements are equivalent:

1. All strongly faithful right $S$-acts are ETF.
2. All strongly faithful finitely generated right $S$-acts are ETF.
3. All strongly faithful cyclic right $S$-acts are ETF.
4. $E(S) = \{1\}$.

Proof. By Lemma [3.3] and Proposition [2.2(2)] it is obvious.

**Theorem 3.5.** Let $S$ be a monoid. Then the following statements are equivalent:

1. All divisible right $S$-acts are ETF.
2. All principally weakly injective right $S$-acts are ETF.
3. All $fg$-weakly injective right $S$-acts are ETF.
4. All weakly injective right $S$-acts are ETF.
5. All injective right $S$-acts are ETF.
6. All cofree right $S$-acts are ETF.
7. All indecomposable right $S$-acts are ETF.
8. All locally cyclic right $S$-acts are ETF.
9. $E(S) = \{1\}$.

Proof. Since cofreeness $\Rightarrow$ injectivity $\Rightarrow$ weak injectivity $\Rightarrow$ $fg$-weak injectivity $\Rightarrow$ principal weak injectivity $\Rightarrow$ divisibility, then implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are obvious. Implications (9) $\Rightarrow$ (1) and (9) $\Rightarrow$ (7) follow from Proposition [2.2(2)]. (7) $\Rightarrow$ (8). It follows from [13, Lemma 3.4]. (6) $\Rightarrow$ (9). Since every right $S$-act can be embedded into a cofree right $S$-act, thus by Proposition [2.2(7)], all right $S$-acts are ETF, and so $E(S) = \{1\}$, by Corollary [3.2]. (8) $\Rightarrow$ (9). All cyclic right $S$-acts are locally cyclic. Thus $E(S) = \{1\}$, by Corollary [3.2].
Here we give a characterization of monoids for which $E$-torsion freeness of their acts implies other properties.

**Theorem 3.6.** Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF right $S$-acts are free.
2. All ETF right $S$-acts are projective generators.
3. All ETF right $S$-acts are generators.
4. All ETF right $S$-acts are faithful.
5. All ETF right $S$-acts are strongly faithful.
6. $S = \{1\}$.

**Proof.** Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (4). It follows from [2, III, 18.1].

Since $\Theta_S$ is an ETF right $S$-act, and $\Theta_S$ is (strongly) faithful if and only if $S = \{1\}$, implications (4), (5) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1), (5). If $S = \{1\}$, then all right $S$-acts are free (strongly faithful).

**Theorem 3.7.** Let $S$ be a monoid with no zero element. Then the following statements are equivalent:

1. All ETF right $S$-acts are torsionless.
2. $S$ contains a left zero.

**Proof.** (1) $\Rightarrow$ (2). Since the right $S$-act $\Theta_S$ is ETF, it follows from [12, Lemma 2.2].

(2) $\Rightarrow$ (1). It follows from Proposition [2, 11] and [12, Proposition 2.10].

Notice that all statements in Theorems 3.6 and 3.7 are also true for cyclic, finitely generated and right Rees factor $S$-acts.

**Theorem 3.8.** Let $S$ be an idempotent monoid. Then the following statements are equivalent:

1. All ETF right $S$-acts are strongly flat.
2. All ETF right $S$-acts are equalizer flat.
3. All ETF right $S$-acts are weakly pullback flat.
4. All ETF right $S$-acts satisfy Condition (P).
5. All ETF right $S$-acts satisfy Condition $(P_E)$.
6. All ETF right $S$-acts are weakly kernel flat.
(7) All ETF right $S$-acts are (WP).

(8) All ETF right $S$-acts are flat.

(9) All ETF right $S$-acts are weakly flat.

(10) All ETF right $S$-acts satisfy Condition $(W)$.

(11) $S$ is right reversible.

(12) $S$ is left collapsible.

**Proof.** (1) ⇔ (3) ⇔ (4). It follows from [14, Page 79].

Implications (4) ⇒ (5) ⇒ (9), (3) ⇒ (6) ⇒ (7) ⇒ (9), (4) ⇒ (8) ⇒ (9) and (1) ⇒ (2) ⇒ (9) are obvious.

(5) ⇔ (9). It follows from [14, Theorem 2.5].

(9) ⇔ (10). Since $S$ is regular, all right $S$-acts are principally weakly flat, and so the result follows from [2, III, 11.4].

(9) ⇒ (11). It follows from Proposition 2.2(1), and [2, III, 11.2].

(11) ⇒ (12). Suppose $e, f \in S$. Since $S$ is right reversible, there exist $g, g' \in S$ such that $ge = g'f$. If $u = ge = g'f$, then $ue = (ge)e = ge = g'f = g'f^2 = (g'f)f = uf$. Thus $S$ is left collapsible.

(12) ⇒ (4). Suppose $A_S$ is ETF and let $as = bt$, for $a, b \in A_S$ and $s, t \in S$. Since by Proposition 2.2(8), $aS = \{a\}$, for any $a \in A_S$, we have $a = b$. Since $S$ is left collapsible, there exists $u \in S$ such that $us = ut$. But, $a = b = au$, and so $A_S$ satisfies Condition $(P)$, as required.

Notice that all statements in theorem above are also true for finitely generated and cyclic right $S$-acts.

### 4 Characterization by $E$-Torsion Freeness of Cyclic Right Acts

In this section we characterize monoids by $E$-torsion freeness of their cyclic right acts.

**Proposition 4.1.** Let $S$ be a monoid and $\rho$ be a right congruence on $S$. Then the following statements are equivalent:

1. $S/\rho$ is ETF.
2. $(\forall s, t \in S)(\forall e \in E(S))(se, te) \in \rho \Rightarrow (s, t) \in \rho$.
3. $(\forall s \in S)(\forall e \in E(S))(se, s) \in \rho$.

**Proof.** It is straightforward. 

Theorem 4.2. Let $\rho$ be a right congruence on $S$. If $S/\rho$ is ETF, then $T = [1]_{\rho}$ is a submonoid of $S$ with $E(S) = E(T)$. The converse is also true when $\rho$ is a congruence or every idempotent of $S$ is central.

Proof. It is obvious that $T$ is a submonoid of $S$ and also $E(T) \subseteq E(S)$. Let $e \in E(S)$. Then $(ee, 1e) = (e, e) \in \rho$, and so $(e, 1) \in \rho$, by Proposition 4.1. Thus $e \in T$, and hence $e \in E(T)$. Suppose $(se, te) \in \rho$, for $s, t \in S$, and $e \in E(S)$. Since $E(S) = E(T)$, we have $(e, 1) \in \rho$. If $\rho$ is a congruence, then $(se, e), (te, t) \in \rho$, and so $(s, t) \in \rho$. If every idempotent of $S$ is central, then $(s, se) = (s, es) \in \rho$ and $(t, te) = (t, et) \in \rho$, and so $(s, t) \in \rho$. Thus in both cases $S/\rho$ is ETF. \hfill $\square$

Corollary 4.3. Let $S$ be an idempotent monoid and $\rho$ be a right congruence on $S$. Then $S/\rho$ is ETF if and only if $S = [1]_{\rho}$.

Proof. Necessity. By Theorem 4.2, we have $S = E(S) = E(T) = T = [1]_{\rho}$. Sufficiency. It is obvious. \hfill $\square$

Theorem 4.4. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All cyclic right $S$-acts are ETF.
(2) All monocyclic right $S$-acts are ETF.
(3) $\rho(x, y) \subseteq \rho(xe, ye)$, for all $x, y \in S$, $e \in E(S)$.
(4) $\rho(x, 1) \subseteq \rho(xe, e)$, for all $x \in S$, $e \in E(S)$.
(5) $\rho(x, e) \subseteq \rho(xe, e)$, for all $x \in S$, $e \in E(S)$.
(6) $\rho(e, f) \subseteq \rho(ef, f)$, for all $e, f \in E(S)$.
(7) $\rho(e, 1) \subseteq \rho(ef, f)$, for all $e, f \in E(S)$.
(8) $\rho(e, f) \subseteq \rho(fe, f)$, for all $e, f \in E(S)$.
(9) $\rho(xe, y) \subseteq \rho(x, y)$, for all $x, y \in S$, $e \in E(S)$.
(10) $\rho(xe, 1) \subseteq \rho(x, 1)$, for all $x \in S$, $e \in E(S)$.
(11) $\rho(xe, 1) \subseteq \rho(x, 1)$, for all $x \in S$, $e \in E(S)$.
(12) $\rho(xe, f) \subseteq \rho(x, f)$, for all $x \in S$, $e, f \in E(S)$.
(13) $\rho(xe, f) \subseteq \rho(x, f)$, for all $x \in S$, $e, f \in E(S)$.
(14) $\rho(xe, f) \subseteq \rho(xe, e)$, for all $x \in S$, $e, f \in E(S)$.
(15) $\rho(xe, f) \subseteq \rho(x, e)$, for all $x \in S$, $e, f \in E(S)$.
(16) $E(S) = \{1\}$.

Proof. It is straightforward. \hfill $\square$
Notice that Theorem 4.4 is also true when inclusions from 3 to 15 be replaced by equality.

Let \( S \) be a monoid and \( s, t \in S \). Set \( F_1 = \{(x, y) \in S \times S \mid \exists e \in E(S), (xe, ye) \in \rho(s, t)\} \), \( F_{i+1} = \{(x, y) \in S \times S \mid \exists e \in E(S), (xe, ye) \in \rho(F_i)\} \), for \( i \in \mathbb{N} \). It can easily be seen that \( F_i \) is reflexive and symmetric, for every \( i \in \mathbb{N} \). Also,

\[
\rho(s, t) \subseteq F_1 \subseteq \rho(F_1) \subseteq F_2 \subseteq \rho(F_2) \subseteq \ldots \subseteq \rho(F_i) \subseteq F_{i+1} \ldots
\]

It is clear that \( \rho_{ETF}(s, t) = \bigcup_{i \in \mathbb{N}} \rho(F_i) \) is a right congruence on \( S \) containing \( (s, t) \).

**Theorem 4.5.** Let \( S \) be a monoid and \( s, t \in S \). Then \( \rho_{ETF}(s, t) \) is the smallest right congruence on \( S \) containing \( (s, t) \), such that \( S/\rho_{ETF}(s, t) \) is ETF.

**Proof.** If \( (xe, ye) \in \rho_{ETF}(s, t) \), for \( x, y \in S \) and \( e \in E(S) \), then there exists \( i \in \mathbb{N} \) such that \( (xe, ye) \in \rho(F_i) \), and so \( (x, y) \in F_{i+1} \). Thus \( (x, y) \in \rho(F_{i+1}) \subseteq \rho_{ETF}(s, t) \), and so \( S/\rho_{ETF}(s, t) \) is ETF by Proposition 4.1.

Let \( \tau \) be a right congruence on \( S \) containing \( (s, t) \), such that \( S/\tau \) is ETF. We show that \( \rho_{ETF}(s, t) \subseteq \tau \). Since \( (s, t) \in \tau \), we have \( \rho(s, t) \subseteq \tau \). If \( (x, y) \in F_1 \), then there exists \( e \in E(S) \) such that \( (xe, ye) \in \rho(s, t) \), and so \( (xe, ye) \in \tau \). Since \( S/\tau \) is ETF, we have \( (x, y) \in \tau \). Thus \( F_1 \subseteq \tau \), and so \( \rho(F_1) \subseteq \tau \). Suppose then that \( \rho(F_i) \subseteq \tau, i \in \mathbb{N} \). If \( (x, y) \in F_{i+1} \), then there exists \( e \in E(S) \) such that \( (xe, ye) \in \rho(F_i) \subseteq \tau \). Since \( S/\tau \) is ETF, \( (x, y) \in \tau \). Hence \( F_{i+1} \subseteq \tau \), and so \( \rho(F_{i+1}) \subseteq \tau \). Thus \( \rho(F_i) \subseteq \tau \), for all \( i \in \mathbb{N} \), and so \( \rho_{ETF}(s, t) \subseteq \tau \).

**Theorem 4.6.** Let \( S \) be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right \( S \)-acts satisfy Condition \( (P) \).
2. All ETF cyclic right \( S \)-acts satisfy Condition \( (P_E) \).
3. For any \( x, y, s, t \in S \), there exist \( u, v \in S \) such that \( (u, x), (v, y) \in \rho_{ETF}(xs, yt) \) and \( us = vt \).
4. For any \( s, t \in S \), there exist \( u, v \in S \) such that \( (u, 1), (v, 1) \in \rho_{ETF}(s, t) \) and \( us = vt \).

**Proof.** (1) \( \Rightarrow \) (2). It is obvious.

(2) \( \Rightarrow \) (3). The cyclic right \( S \)-act \( S/\rho_{ETF}(xs, yt) \) is ETF, and so it satisfies Condition \( (P_E) \). Thus by [15, Theorem 2.5], there exist \( u, v \in S \) and \( e, f \in E(S) \) such that \( us = vt, es = s, ft = t, (ue, xe), (yf, vf) \in \rho_{ETF}(xs, yt) \). Thus by Proposition 4.1, \( (u, x), (v, y) \in \rho_{ETF}(xs, yt) \), as required.

(3) \( \Rightarrow \) (4). It is sufficient to take \( x = y = 1 \).

(4) \( \Rightarrow \) (1). Suppose \( \tau \) is a right congruence on \( S \), such that \( S/\tau \) is ETF and let \( (s, t) \in \tau \). Then by assumption, there exist \( u, v \in S \) such that \( us = vt \) and \( (u, 1), (v, 1) \in \rho_{ETF}(s, t) \). By Theorem 4.5, \( \rho_{ETF}(s, t) \subseteq \tau \), and so \( (u, 1), (v, 1) \in \tau \). Thus \( S/\tau \) satisfies Condition \( (P) \), by [2, III, 13.4].
Theorem 4.7. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts satisfy Condition $(P')$.
2. For any $x, y, z, t, t' \in S$, the equality $tz = t'z$ implies that there exist $u, v \in S$ such that $ut = vt'$ and $(u, x), (v, y) \in \rho_{ETF}(xt, yt')$.

Proof. Using [16, Theorem 3.1] and Theorem 4.5, it is similar to that of Theorem 4.6.

Theorem 4.8. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts satisfy Condition $(E)$.
2. For any $s, t \in S$, there exists $u \in S$ such that $us = ut$, and $(u, 1) \in \rho_{ETF}(s, t)$.

Proof. Using [2, III, 14.8] and Theorem 4.5, it is similar to that of Theorem 4.6.

Theorem 4.9. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts satisfy Condition $(E')$.
2. For any $s, t, z \in S$, the equality $sz = tz$ implies that there exists $u \in S$ such that $us = ut$ and $(u, 1) \in \rho_{ETF}(s, t)$.

Proof. It follows from Theorem 4.5 and a similar argument as in the proof of Theorem 4.6.

Theorem 4.10. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts satisfy Condition $(E'P)$.
2. For any $x, y, z \in S$, the equality $xz = yz$ implies that there exist $u, v \in S$ such that $ux = vy$ and $(u, 1), (v, 1) \in \rho_{ETF}(x, y)$.
3. For any $x, t, t', z \in S$, the equality $tz = t'z$ implies that there exist $u, v \in S$ such that $ut = vt'$ and $(u, x), (v, x) \in \rho_{ETF}(xt, xt')$.

Proof. It follows from [17, Theorem 2.10], Theorem 4.5, and a similar argument as in the proof of Theorem 4.6.

Theorem 4.11. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts are principally weakly flat.
2. For any $u, v, s \in S$, $(u, v) \in (\rho_{ETF}(us, vs) \vee \ker \rho_s)$.

Proof. $(1) \Rightarrow (2)$. Suppose $u, v, s \in S$. Then the cyclic right $S$-act $S/\rho_{ETF}(us, vs)$ is ETF, and so it is principally weakly flat. Since $(us, vs) \in \rho_{ETF}(us, vs)$ by [2, III, 10.7], we have $(u, v) \in (\rho_{ETF}(us, vs) \vee \ker \rho_s)$.

$(2) \Rightarrow (1)$. Suppose $\tau$ is a right congruence on $S$, such that $S/\tau$ is ETF and let $(us, vs) \in \tau$. By Theorem 4.5, $(\rho_{ETF}(us, vs) \subseteq \tau$. By assumption, $(u, v) \in (\rho_{ETF}(us, vs) \vee \ker \rho_s)$, and so $(u, v) \in (\tau \vee \ker \rho_s)$. Thus $S/\tau$ is principally weakly flat, by [2, III, 10.7].
Theorem 4.12. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts are weakly flat.

2. For any $s, t \in S$, there exist $u, v \in S$ such that $us = vt$, $(u, 1) \in (\rho_{ETF}(s, t) \lor ker\rho_s)$ and $(v, 1) \in (\rho_{ETF}(s, t) \lor ker\rho_t)$.

Proof. $(1) \Rightarrow (2)$. The cyclic right $S$-act $S/\rho_{ETF}(s, t)$ is ETF, and so it is weakly flat. Thus by [2, III, 11.5], there exist $u, v \in S$ such that $us = vt$, $(u, 1) \in (\rho_{ETF}(s, t) \lor ker\rho_s)$ and $(v, 1) \in (\rho_{ETF}(s, t) \lor ker\rho_t)$.

$(2) \Rightarrow (1)$. Suppose $\tau$ is a right congruence on $S$, such that $S/\tau$ is ETF and let $(s, t) \in \tau$. By Theorem 4.5, $\rho_{ETF}(s, t) \subseteq \tau$ and by assumption, there exist $u, v \in S$ such that $us = vt$, $(u, 1) \in (\rho_{ETF}(s, t) \lor ker\rho_s)$ and $(v, 1) \in (\rho_{ETF}(s, t) \lor ker\rho_t)$. Thus $(u, 1) \in (\tau \lor ker\rho_s)$ and $(v, 1) \in (\tau \lor ker\rho_t)$, and so $S/\tau$ is weakly flat, by [2, III, 11.5].

Theorem 4.13. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts satisfy Condition (PWP).

2. All ETF cyclic right $S$-acts satisfy Condition (PWP$_E$).

3. For any $x, y, t \in S$, there exist $u, v \in S$ such that $ut = vt$ and $(u, x), (v, y) \in \rho_{ETF}(xt, yt)$.

Proof. Using [9, Theorem 3.7], [5, Proposition 7] and Theorem 4.5, it is similar to that of Theorem 4.6.

Lemma 4.14. Let $S$ be a left PP monoid. Then the following statements are equivalent:

1. For any $x, y, t \in S$, $(x, y) \in (\rho_{ETF}(xt, yt) \lor ker\rho_t)$.

2. For any $x, y, t \in S$, there exist $u, v \in S$ such that $ut = vt$ and $(u, x), (v, y) \in \rho_{ETF}(xt, yt)$.

3. For any $x, y, t \in S$, $(x, y) \in \rho_{ETF}(xt, yt)$.

Proof. $(1) \Leftrightarrow (2)$. It follows from [9, Theorem 2.5], Theorem 4.11 and Theorem 4.13.

$(2) \Rightarrow (3)$. Let $x, y, t \in S$. By assumption there exist $u, v \in S$ such that $ut = vt$ and $(u, x), (v, y) \in \rho_{ETF}(xt, yt)$. Since $S$ is left PP, there exists $e \in E(S)$ such that $ker\rho_s = ker\rho_e$. Thus $we = we$, and so $(xe, ye) \in \rho_{ETF}(xt, yt)$. Hence $(x, y) \in \rho_{ETF}(xt, yt)$, by Proposition 4.1 and Theorem 4.5.

$(3) \Rightarrow (1)$. It is obvious.

Theorem 4.15. Let $S$ be a left PP monoid. Then the following statements are equivalent:

1. All ETF cyclic right $S$-acts satisfy Condition (PWP).

2. All ETF cyclic right $S$-acts satisfy Condition (PWP$_E$).
(3) All ETF cyclic right $S$-acts are principally weakly flat.

(4) $S$ acts injectively on every ETF cyclic right $S$-acts.

(5) For any $x, y, t \in S$, $(x, y) \in \rho_{ETF}(xt, yt)$.

Proof. (1) $\iff$ (2). It follows from Theorem 4.13.

(2) $\Rightarrow$ (3). It follows from [9, Theorem 2.5].

(3) $\Rightarrow$ (5). It follows from Theorem 4.11 and Lemma 4.14.

(4) $\Rightarrow$ (5). Suppose $x, y, t \in S$. Then the cyclic right $S$-act $S/\rho_{ETF}(xt, yt)$ is ETF, and so $S$ acts injectively on $S/\rho_{ETF}(xt, yt)$. $(xt, yt) \in \rho_{ETF}(xt, yt)$ implies that $[x]_{\rho_{ETF}(xt, yt)}t = [y]_{\rho_{ETF}(xt, yt)}t$, and so $[x]_{\rho_{ETF}(xt, yt)} = [y]_{\rho_{ETF}(xt, yt)}$. Thus $(x, y) \in \rho_{ETF}(xt, yt)$.

(5) $\Rightarrow$ (4). Suppose $\tau$ is a right congruence on $S$, such that $S/\tau$ is ETF. Let $[x]_{\tau}t = [y]_{\tau}t$, $x, y, t \in S$. By Theorem 4.5 and by the assumption we have $(x, y) \in \rho_{ETF}(xt, yt) \subseteq \tau$. Thus $[x]_{\tau} = [y]_{\tau}$ as required.

Theorem 4.16. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All ETF cyclic right $S$-acts are torsion free.

(2) For any $s, t \in S$, $\rho_{TF}(s, t) \subseteq \rho_{ETF}(s, t)$.

(3) For any $s, t \in S$ and $c \in S$ right cancellable, $(s, t) \subseteq \rho_{ETF}(sc, tc)$.

Proof. (1) $\Rightarrow$ (2). Suppose $s, t \in S$. Then the cyclic right $S$-act $S/\rho_{ETF}(s, t)$ is ETF, and so it is torsion free. Thus $\rho_{TF}(s, t) \subseteq \rho_{ETF}(s, t)$, by the proof of Lemma 3.31 of [4].

(2) $\Rightarrow$ (3). Suppose $s, t \in S$ and $c \in S$ right cancellable. Then $\rho_{TF}(s, t) \subseteq \rho_{ETF}(sc, tc)$, by [4] Lemma 3.31 and [2, III, 8.4].

(3) $\Rightarrow$ (1). Suppose $\tau$ is a right congruence on $S$, such that $S/\tau$ is ETF and let $(sc, tc) \in \tau$, $s, t \in S$ and $c \in S$ right cancellable. Then by Theorem 4.5 and by the assumption we have $(s, t) \subseteq \rho_{ETF}(sc, tc) \subseteq \tau$. Thus $S/\tau$ is torsion free, by [2, III, 8.4].

Theorem 4.17. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All ETF cyclic right $S$-acts are weakly pullback flat.

(2) $S$ satisfies the following Conditions:

(a) For any $s, t, z \in S$, the equality $sz = tz$ implies that there exists $u \in S$ such that $us = ut$ and $(u, 1) \in \rho_{ETF}(s, t)$.

(b) For any $s, t \in S$, there exist $u, v \in S$ such that $us = vt$ and $(u, 1), (v, 1) \in \rho_{ETF}(s, t)$.

Proof. It follows from [3, Theorem 21], Theorem 4.6 and Theorem 4.9.

Theorem 4.18. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All torsion free cyclic right $S$-acts are ETF.
Proof. It is similar to the proof of Theorem 4.16.

5 Characterization by \( E \)-torsion freeness of mono-
cyclic right acts

In this section we characterize monoids by \( E \)-torsion freeness of their mono-
cyclic right acts.

Lemma 5.1. Let \( S \) be a monoid, \( w, t \in S \) and \( w \neq t \). Then \( \rho(wt, t) = \rho(w, 1) \) if
and only if \( t \) is right invertible.

Proof. Suppose \( \rho(wt, t) = \rho(w, 1) \). Then by [2, III, 8.5], there exist \( m, n \in \mathbb{N} \cup \{0\} \)
such that \( w^m w = w^n 1 = w^n \) and \( w^i w \in tS \), whenever \( 0 \leq i < m \), and \( w^j \in tS \),
whenever \( 0 \leq j < n \). If \( n \geq 1 \), then \( 1 = w^n \in tS \), and so \( t \) is right invertible. If
\( n = 0 \), then \( n \geq 1 \), since \( w \neq 1 \). Thus \( w \in tS \), and so \( 1 = w^{m+1} \in tS \), that is, \( t \) is
right invertible. The converse is obvious.

Theorem 5.2. Let \( S \) be a monoid and \( w, e^2 = e \in S \). Then \( S/\rho(we, e) \) is ETF if
and only if \( e = 1 \) and \( w^m x f = w^n y f \), for \( x, y, f^2 = f \in S \), \( m, n \in \mathbb{N} \cup \{0\} \), implies
that \( w^p x = w^q y \), for some \( p, q \in \mathbb{N} \cup \{0\} \).

Proof. Let \( \rho = \rho(we, e) \). Necessity. If \( we = e \), then \( S/\rho = S/\Delta_S \cong S_S \), and so we are done by Proposition
2.2. Thus we suppose \( we \neq e \). Since \( (we, e) \in \rho \), we have by Proposition 4.1 that \( (w, 1) \in \rho \). Since \( (we, e) \in \rho(we, 1) \), we have \( \rho = \rho(we, 1) \). Thus \( e = 1 \), by
Lemma 5.1. Suppose now that \( w^m x f = w^n y f \), for \( x, y, f^2 = f \in S \) and \( m, n \in \mathbb{N} \cup \{0\} \). Then \( (xf, yf) \in \rho \), by [2, III, 8.7], and so \( (x, y) \in \rho \), by Proposition 4.1.
Since \( \rho = \rho(we, 1) \), we have by [2, III, 8.7], that \( w^p x = w^q y \), for some \( p, q \in \mathbb{N} \cup \{0\} \).
Sufficiency. Suppose \( (sf, tf) \in \rho \). Then by [2, III, 8.7], \( w^m sf = w^n tf \), for some \( m, n \in \mathbb{N} \cup \{0\} \). Thus by assumption, \( w^p s = w^q t \), for some \( p, q \in \mathbb{N} \cup \{0\} \). Again by [2, III, 8.7], \( (s, t) \in \rho \), and so the result follows from Proposition 4.1.

Theorem 5.3. Let \( S \) be a monoid. Then all ETF right \( S \)-acts of the form
\( S/\rho(we, e) \) satisfy Condition (P).

Proof. It follows from Theorem 5.2 and [2, III, 13.8] and [2, III, 13.5].

Theorem 5.4. Let \( S \) be a monoid. Then the following statements are equivalent:

1. All monocyclic right \( S \)-acts of the form \( S/\rho(we, e) \), \( w, e^2 = e \in S \), \( we \neq e \),
satisfying Condition (P) are ETF.
(2) For every $1 \neq w \in S$, if there exist $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$ such that $w^mxf = w^nyf$, then there exist $p, q \in \mathbb{N} \cup \{0\}$ such that $w^px = w^qy$.

Proof. $(1) \Rightarrow (2)$. Let $w \in S$, with $w \neq 1$. Then $S/\rho(w, 1)$ satisfies Condition (P), by [2, III, 13.8]. Thus $S/\rho(w, 1)$ is ETF, and the result follows from Theorem 5.2.

$(2) \Rightarrow (1)$. Suppose the right $S$-act $S/\rho(we, e)$, $we \neq e$, satisfies Condition (P). Then by [2, III, 13.8], there exists $a \in S$ such that $\rho(we, e) = \rho(a, 1)$. Since $(we, e) \in \rho(a, 1)$, by [2, III, 8.7], there exist $m, n \in \mathbb{N} \cup \{0\}$ such that $a^mwe = a^n$. Since $we \neq e$, we have $a \neq 1$. Thus by assumption there exist $p, q \in \mathbb{N} \cup \{0\}$ such that $a^pw = a^q$. Again $(w, 1) \in \rho(a, 1) = \rho(we, e)$ by [2, III, 8.7], and so $\rho(we, e) = \rho(w, 1)$. Now the result follows from Lemma 5.1 and Theorem 5.2.

Theorem 5.5. Let $S$ be a monoid. Then the following statements are equivalent:

$(1)$ All projective monocyclic right $S$-acts of the form $S/\rho(w, 1), 1 \neq w \in S$, are ETF.

$(2)$ All monocyclic right $S$-acts of the form $S/\rho(w, 1), 1 \neq w \in S$, satisfying Condition (E) are ETF.

$(3)$ If $1 \neq w$ is aperiodic, then the equality $w^nx = w^nyf$, for $x, y, f^2 = f \in S$ and $n \in \mathbb{N}$, implies $w^nx = w^ny$.

Proof. $(1) \Leftrightarrow (2)$. It follows from [2, III, 17.14].

$(2) \Rightarrow (3)$. Suppose for $1 \neq w \in S$, there exists $n \in \mathbb{N}$ such that $w^{n+1} = w^n$. Then $S/\rho(w, 1)$ satisfies Condition (E), by [2, III, 17.14], and so it is ETF. Now the result follows from Theorem 5.2.

$(3) \Rightarrow (2)$. Suppose the right $S$-act $S/\rho(w, 1), w \neq 1$, satisfies Condition (E). Then by [2, III, 17.14], there exists $n \in \mathbb{N}$ such that $w^{n+1} = w^n$. If $w^nx = w^nyf$, for $x, y, f^2 = f \in S$ and $k, j \in \mathbb{N} \cup \{0\}$, then $w^nx = w^nyf$, and so by assumption, $w^nx = w^ny$. Now the result follows from Theorem 5.2.

Now we consider monoids over which $E$-torsion freeness implies projectivity and Condition (E).

Theorem 5.6. Let $S$ be a monoid. Then the following statements are equivalent:

$(1)$ All ETF monocyclic right $S$-acts of the form $S/\rho(w, 1)$, are projective.

$(2)$ All ETF monocyclic right $S$-acts of the form $S/\rho(w, 1)$, satisfy Condition (E).

$(3)$ Every $w \in S$ is either aperiodic or there exist $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$ such that $w^mxf = w^nyf$ and $w^px \neq w^qy$, for all $p, q \in \mathbb{N} \cup \{0\}$.

Proof. $(1) \Leftrightarrow (2)$. It follows from [2, III, 17.14].

$(2) \Rightarrow (3)$. If $w = 1$, then we are done. Suppose that $1 \neq w \in S$. If for all $x, y, f^2 = f \in S$ and $m, n \in \mathbb{N} \cup \{0\}$, $w^mxf = w^nyf$ implies the existence of $p, q \in \mathbb{N} \cup \{0\}$ such that $w^px = w^qy$, then by Theorem 5.2 $S/\rho(w, 1)$ is ETF.
Thus $S/\rho(w,1)$ satisfies Condition (E), and so $w$ is aperiodic, by \cite{2} III, 14.9.

(3) $\Rightarrow$ (2). Suppose the right $S$-act $S/\rho(w,1)$, $w \in S$ is ETF. If $w = 1$, then $S/\rho(w,1) = S/\Delta_S \cong S$ satisfies Condition (E). Thus we suppose $w \neq 1$. By Theorem 5.2 the equality $w^mxf = w^nyf$, for all $x,y,f \in S$ and $m,n \in \mathbb{N} \cup \{0\}$, implies the existence of $p,q \in \mathbb{N} \cup \{0\}$ such that $w^px = w^qy$. Thus $w$ is aperiodic and the result follows from \cite{2} III, 14.9.

6 Characterization by $E$-Torsion Freeness of Right Rees Factor Acts

In this section we characterize monoids by $E$-torsion freeness of their right Rees factor acts. First of all we give a characterization of monoids over which all right Rees factor acts are ETF and also monoids over which all ETF right Rees factor acts have some other properties. Then we give a characterization of monoids for which right Rees factor acts with other properties are ETF. We recall that for a right ideal $K_S$ of $S$, the Rees congruence $\rho_K$ is defined by $(a,b) \in \rho_K$ if $a,b \in K_S$ or $a = b$ and the resulting factor act is called the Rees factor act and is denoted by $S/K_S$.

Theorem 6.1. Let $S$ be a monoid and $K_S$ be a right ideal of $S$. Then $S/K_S$ is ETF if and only if $K_S = S$ or $E(S) = \{1\}$.

Proof. Necessity. Suppose $S/K_S$ is ETF, $K_S \neq S$ and let $e \in E(S)$. Then $[1]_{\rho_K} e = [e]_{\rho_K} e$, and so $[1]_{\rho_K} = [e]_{\rho_K}$. Thus $e = 1$, and so $E(S) = \{1\}$ as required. Sufficiency. It follows from Proposition 2.2(1),(2).

Theorem 6.2. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All right Rees factor $S$-acts are ETF.

(2) All right Rees factor $S$-acts of the form $S/sS$, $s \in S$ are ETF.

(3) All right Rees factor $S$-acts of the form $S/sS$, $s \in S$ is regular, are ETF.

(4) All right Rees factor $S$-acts of the form $S/eS$, $e \in E(S)$, are ETF.

(5) $E(S) = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.

(4) $\Rightarrow$ (5). Let $e \in E(S)$. Then by assumption $S/eS$ is ETF, and so by Theorem 6.1 $eS = S$ or $E(S) = \{1\}$. these imply that $E(S) = \{1\}$.

(5) $\Rightarrow$ (1). It follows from Proposition 2.2(2).

Theorem 6.3. Let $S$ be a monoid and $(U)$ be a property of acts. Then the following statements are equivalent:

(1) All ETF right Rees factor $S$-acts satisfy $(U)$. 
(2) $\Theta_S$ satisfies $(U)$ and $E(S) = \{1\}$ implies that all right Rees factor $S$-acts satisfy $(U)$.

**Proof.** (1) $\Rightarrow$ (2). By Proposition 2.2(1), $\Theta_S \cong S/S_S$ satisfies $(U)$. If $E(S) = \{1\}$, then by Theorem 6.2 and assumption all right Rees factor $S$-acts satisfy $(U)$.

(2) $\Rightarrow$ (1). Suppose the right Rees factor $S$-act $S/K_S$ is ETF. Then either $K_S = S$ or $E(S) = \{1\}$, by Theorem 6.1. If $K_S = S$, then $S/K_S = S/S_S \cong \Theta_S$ satisfies Condition $(U)$. If $E(S) = \{1\}$, then by assumption, all right Rees factor $S$-acts satisfy $(U)$, and so all ETF right Rees factor $S$-acts satisfy $(U)$.

**Theorem 6.4.** Let $S$ be a monoid. Then the following statements are equivalent:

(1) All ETF right Rees factor $S$-acts are torsion free.

(2) $E(S) \neq \{1\}$ or else, every right cancellable element of $S$ is right invertible.

**Proof.** It follows from Theorem 6.3 [2, III, 8.2] and [2, IV, 6.1].

**Theorem 6.5.** Let $S$ be a monoid. Then the following statements are equivalent:

(1) All ETF right Rees factor $S$-acts are principally weakly flat.

(2) All ETF right Rees factor $S$-acts satisfy Condition $(PWP)$.

(3) All ETF right Rees factor $S$-acts satisfy Condition $(PWP_E)$.

(4) $E(S) \neq \{1\}$ or else, $S$ is a group.

**Proof.** (1) $\Leftrightarrow$ (4). It follows from Theorem 6.3 [2, III, 10.2], [2, IV, 6.6], and [1, II, Exercise 11].

(2) $\Leftrightarrow$ (4). It follows from Theorem 6.3 [5, Corollary 11] and [3, Proposition 9].

(3) $\Leftrightarrow$ (4). It follows from Theorem 6.3 and [9, Theorem 3.1].

**Theorem 6.6.** Let $S$ be a monoid. Then the following statements are equivalent:

(1) All ETF right Rees factor $S$-acts are flat.

(2) All ETF right Rees factor $S$-acts are weakly flat.

(3) All ETF right Rees factor $S$-acts satisfy Condition $(WP)$.

(4) All ETF right Rees factor $S$-acts satisfy Condition $(P)$.

(5) All ETF right Rees factor $S$-acts satisfy Condition $(P_E)$.

(6) All ETF right Rees factor $S$-acts are $P$-regular.

(7) $S$ is right reversible and $E(S) = \{1\}$ implies that $S$ is a group.

**Proof.** (1) $\Leftrightarrow$ (2). It follows from [2, III, 12.17].

(2) $\Leftrightarrow$ (7). It follows from Theorem 6.3 [2, III, 11.2] and [2, IV, 7.3].

(3) $\Leftrightarrow$ (7). It follows from Theorem 6.3 [5, Corollary 18] and [3, Proposition 14].

(4) $\Leftrightarrow$ (7). It follows from Theorem 6.3 [5, Corollary 18] and [2, IV, 9.9].

(5) $\Leftrightarrow$ (7). It follows from Theorem 6.3 and [15, Theorem 3.1].

(6) $\Leftrightarrow$ (7). It follows from Theorem 6.3 [10, Theorem 2.1(1)] and [10, Theorem 2.3].
Theorem 6.7. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF right Rees factor $S$-acts satisfy Condition $(E)$.
2. All ETF right Rees factor $S$-acts are pullback flat.
3. All ETF right Rees factor $S$-acts are equalizer flat.
4. All ETF right Rees factor $S$-acts are strongly flat.
5. $S$ is left collapsible and $E(S) = \{1\}$ implies that $S = \{1\}$.

Proof. Implications $(1) \iff (2)$, $(2) \iff (3)$ and $(3) \iff (4)$ are obvious, by [2, III, 16.7].

$(4) \iff (5)$. By Theorem 6.3, all ETF right Rees factor $S$-acts are strongly flat if and only if $\Theta_S$ is strongly flat and $E(S) = \{1\}$ implies that all right Rees factor $S$-acts are strongly flat. By [2, III, 14.3] and [2, IV, 11.13], all ETF right Rees factor $S$-acts are strongly flat if and only if $S$ is left collapsible and $E(S) = \{1\}$ implies that $S = \{1\}$. \hfill $\Box$

Theorem 6.8. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF right Rees factor $S$-acts are regular.
2. All ETF right Rees factor $S$-acts are strongly $(P)$-cyclic.
3. All ETF right Rees factor $S$-acts are projective.
4. $S$ contains a left zero.

Proof. $(1) \Rightarrow (2)$. It follows from [11, Theorem 2.1].

$(2) \Rightarrow (3)$. It follows from [11, Corollary 3.10].

$(3) \Rightarrow (4)$. By Proposition 2.2(1), the right Rees factor $S$-act $S/S \cong \Theta_S$ is ETF, and so it is projective. Thus by [2, III, 17.2], $S$ contains a left zero element.

$(4) \Rightarrow (1)$. Suppose the right Rees factor $S$-act $S/K_S$ is ETF. By Theorem 6.1, $K_S = S$ or $E(S) = \{1\}$. If $K_S = S$, then $S/K_S = S/S \cong \Theta_S$ is regular, by [2, III, 19.4(4)]. If $E(S) = \{1\}$, then $S = \{1\}$, and so all right Rees factor $S$-acts are regular. \hfill $\Box$

Notice that all statements in Theorem 6.8 are also true for (cyclic) right $S$-acts.

Theorem 6.9. Let $S$ be a monoid. Then the following statements are equivalent:

1. All ETF right Rees factor $S$-acts are weakly pullback flat.
2. All ETF right Rees factor $S$-acts are weakly kernel flat.
3. $S$ is right reversible, weakly left collapsible and $E(S) = \{1\}$ implies that $S$ is a group.

Proof. It follows from Theorem 6.3 [10, Corollary 3.10] and [5, Theorem 20]. \hfill $\Box$
Theorem 6.10. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All ETF right Rees factor $S$-acts are principally weakly kernel flat.

(2) All ETF right Rees factor $S$-acts are translation kernel flat.

(3) $\ker \rho_z$ is connected as a left $S$-act, for every $z \in S$ and $E(S) = \{1\}$ implies that $S$ is a group.

Proof. It follows from Theorem 6.3, [3, Proposition 7] and [3, Theorem 20].

Theorem 6.11. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All ETF right Rees factor $S$-acts satisfy Condition $(P')$.

(2) $S$ is weakly right reversible and $E(S) = \{1\}$ implies that $S$ is a group.

Proof. It follows from Theorem 6.3, [16, Corollary 4.4] and [16, Thorem 4.18].

Now we give a characterization of monoids for which right Rees factor acts with other properties are ETF.

Theorem 6.12. Let $S$ be a monoid. Then the following statements are equivalent:

(1) All right Rees factor $S$-acts satisfying Condition $(P)$ are ETF.

(2) All right Rees factor $S$-acts satisfying Condition $(E)$ are ETF.

(3) All pullback flat right Rees factor $S$-acts are ETF.

(4) All equalizer flat right Rees factor $S$-acts are ETF.

(5) All strongly flat right Rees factor $S$-acts are ETF.

(6) All projective right Rees factor $S$-acts are ETF.

(7) All free right Rees factor $S$-acts are ETF.

(8) $S$ contains no left zero element or $S = \{1\}$.

Proof. (1) $\Rightarrow$ (2). It follows from [2, III, 14.7].
Implications (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follow from [2, III, 16.7].
Implications (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are obvious.

(7) $\Rightarrow$ (8). Let $s \in S$ be a left zero. Then the right Rees factor $S$-act $S/sS \cong S_S$ is free, and so it is ETF, by assumption. Thus by Proposition 2.2(3), $E(S) = \{1\}$, and so $s = 1$, that is, $S = \{1\}$.

(8) $\Rightarrow$ (1). Suppose the right Rees factor $S$-act $S/K_S$ satisfies Condition $(P)$. If $K_S = S$, then by Proposition 2.2(1) we are done. Thus we suppose that $K_S \neq S$.
Then by [2, III, 13.9], $|K_S| = 1$, and so $S$ contains a left zero element. Thus, $S = \{1\}$, and so $K_S = S$, which is a contradiction.

Theorem 6.13. Let $S$ be a monoid. Then the following statements are equivalent:
(1) All weakly flat right Rees factor $S$-acts are ETF.
(2) All flat right Rees factor $S$-acts are ETF.
(3) $E(S) = \{1\}$ or $S$ is not right reversible.

Proof. (1) $\Leftrightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (3). Suppose $E(S) \neq \{1\}$. $S$ is right reversible and let $e \in E(S) \setminus \{1\}$. By \cite[III, 12.17]{2}, the right Rees factor $S/(eS)_S$ is flat. Thus $eS = S$, by Theorem 6.1, which is a contradiction.
(3) $\Rightarrow$ (1). It follows from \cite[III, 12.17]{2} and Proposition 2.2(2).

Theorem 6.14. Let $S$ be a monoid. Then the following statements are equivalent:
(1) All torsion free right Rees factor $S$-acts are ETF.
(2) All principally weakly flat right Rees factor $S$-acts are ETF.
(3) $E(S) = \{1\}$.

Proof. (1) $\Rightarrow$ (2). It is obvious.
(2) $\Rightarrow$ (3). Suppose $e \in E(S)$. The right Rees factor $S/(eS)_S$ is principally weakly flat, and so $E(S) = \{1\}$ or $eS = S$, by Theorem 6.1. In each case, $E(S) = \{1\}$.
(3) $\Rightarrow$ (1). It follows from Proposition 2.2(2).

Theorem 6.15. Let $S$ be a monoid. Then the following statements are equivalent:
(1) All regular right Rees factor $S$-acts are ETF.
(2) $S = \{1\}$ or $S$ contains no left zero element or there exists $z \in S$ such that $\ker\lambda z \neq \ker\lambda e$, for every $e \in E(S)$.

Proof. (1) $\Rightarrow$ (2). It follows from \cite[III, 19.6]{2}, \cite[III, 17.16]{2} and Theorem 6.1.
(2) $\Rightarrow$ (1). It follows from \cite[III, 19.6]{2}, \cite[III, 17.16]{2} and Proposition 2.2(2).

Theorem 6.16. Let $S$ be a monoid. Then the following statements are equivalent:
(1) All $P$-regular right Rees factor $S$-acts are ETF.
(2) All strongly $(P)$-cyclic right Rees factor $S$-acts are ETF.
(3) $S = \{1\}$ or $S$ contains no left zero element or $S$ is not right PCP.

Proof. Since strong $(P)$-cyclic implies $P$-regularity, (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3). It follows from \cite[Theorem 3.1]{11} and Theorem 6.1.
(3) $\Rightarrow$ (1). It follows from \cite[Theorem 3.1]{10} and Theorem 6.2.

Theorem 6.17. Let $S$ be a monoid. Then the following statements are equivalent:
(1) All right Rees factor $S$-acts satisfying Condition $(P')$ are ETF.
(2) $E(S) = \{1\}$ or $S$ has no left stabilizing and completely left annihilating proper right ideal.
Proof. (1) ⇒ (2). It follows from [16, Theorem 4.3] and Theorem 6.1.
(2) ⇒ (1). It follows from [16, Theorem 4.3] and Theorem 6.2.

**Theorem 6.18.** Let $S$ be a monoid. Then the following statements are equivalent:

1. All right Rees factor $S$-acts satisfying Condition (PWP) are ETF.
2. $E(S) = \{1\}$ or $S$ has no left stabilizing and left annihilating proper right ideal.

Proof. (1) ⇒ (2). It follows from [5, Theorem 10] and Theorem 6.1.
(2) ⇒ (1). It follows from [5, Theorem 10] and Theorem 6.2.

**Theorem 6.19.** Let $S$ be a monoid. Then the following statements are equivalent:

1. All right Rees factor $S$-acts satisfying Condition (WP) are ETF.
2. $E(S) = \{1\}$ or $S$ is not right reversible or $S$ has no left stabilizing and strongly left annihilating proper right ideal.

Proof. (1) ⇒ (2). It follows from [5, Theorem 17] and Theorem 6.1.
(2) ⇒ (1). It follows from [5, Theorem 17] and Theorem 6.2.

**Theorem 6.20.** Let $S$ be a monoid. Then the following statements are equivalent:

1. All right Rees factor $S$-acts satisfying Condition (PE) are ETF.
2. $E(S) = \{1\}$ or $S$ is not right reversible or $S$ has no proper $P_E$-left annihilating right ideal.

Proof. (1) ⇒ (2). It follows from [15, Theorem 3.5] and Theorem 6.1.
(2) ⇒ (1). It follows from [15, Theorem 3.5] and Theorem 6.2.

**Theorem 6.21.** Let $S$ be a monoid. Then the following statements are equivalent:

1. All right Rees factor $S$-acts satisfying Condition (PWP) are ETF.
2. $E(S) = \{1\}$ or $S$ has no left stabilizing and $E$-left annihilating proper right ideal.

Proof. (1) ⇒ (2). It follows from [9, Theorem 4.2] and Theorem 6.1.
(2) ⇒ (1). It follows from [9, Theorem 4.2] and Theorem 6.2.

**References**


(Received 18 July 2013)
(Accepted 8 April 2014)