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A quadratic functional equation and its generalized Hyers-Ulam Rassias stability

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Abstract : We study the general solution of the quadratic functional equation

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+16f(x)$$

and investigate its generalized Hyers-Ulam-Rassias stability.

Keywords : Functional Equation; Quadratic Functional Equation; Stability; Hyers-Ulam-Rassias Stability
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1 Introduction

In 1940, S.M. Ulam [12] proposed the Ulam stability problem of linear mappings. In the next year, D.H. Hyers [5] considered the case of approximately additive mappings $f : E \to E'$ where E and E' are Banach spaces and f satisfies inequality $\|f(x+y)-f(x)-f(y)\| \le \varepsilon$ for all $x, y \in E$. It was shown that the limit $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all $x \in E$ and that L is the unique additive mapping satisfying $\|f(x)-L(x)\| \le \varepsilon$. In recent years, a number of authors [7, 6, 9] have investigated the stability of linear mappings in various forms.

In this paper, we will study a quadratic functional equation and will investigate its generalized Hyers-Ulam-Rassias stability.

2 The general solution

Theorem 2.1. *Let X and Y be a real vector space. A function $f : X \to Y$ satisfies*

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+16f(x) \quad \forall x,y \in X \tag{2.1}$$

if and only if it satisfies

$$f(x+y)+f(x-y)=2f(x)+2f(y) \quad \forall x,y \in X. \tag{2.2}$$

Proof. Suppose a function $f : X \to Y$ satisfies (2.1). Putting $(x, y) = (0, 0)$ in (2.1), we get $f(0) = 0$. Setting $y = x$ in (2.1), we obtain

$$f(4x) = 16f(x) \tag{2.3}$$

for all $x \in X$. Putting $(x, y) = (x, x - y)$ in (2.1), we have

$$f(4x - y) + f(2x + y) = f(2x - y) + f(y) + 16f(x). \tag{2.4}$$

If we reverse the sign of y in (2.4), then add the resulting equation to (2.4), we will have

$$f(4x - y) + f(4x + y) = f(y) + f(-y) + 32f(x). \tag{2.5}$$

Replacing y with $4y$ in (2.5) and using (2.3), we will be left with

$$f(x + y) + f(x - y) = f(y) + f(-y) + 2f(x). \tag{2.6}$$

Putting $y = x$ in (2.6), we get

$$f(2x) = 3f(x) + f(-x). \tag{2.7}$$

Reversing the sign of x in (2.7) gives us $f(-2x) = 3f(-x) + f(x)$. Replacing x with $2x$ in (2.7) and taking into account (2.3), we obtian

$$16f(x) = 3f(2x) + f(-2x) = 3\big(3f(x) + f(-x)\big) + \big(3f(-x) + f(x)\big),$$

which simplifies to $f(x) = f(-x)$ for all $x \in X$. Hence, (2.6) reduces to (2.2).

Suppose a function $f : X \to Y$ satisfies (2.2). It is known [4] that f possesses a quadratic property; i.e., $f(nx) = n^2 f(x)$ for all integers n and for all $x \in X$. Thus,

$$f(3x + y) + f(3x - y) = 2f(3x) + 2f(y) = 18f(x) + 2f(y)$$

and we can see that (2.1) immediately follows. □

3 The Generalized Hyers-Ulam-Rassias Stability

Theorem 3.1. *Let X be a real vector space and let Y be a Banach space. Let $\phi : X^2 \to [0, \infty)$ be a function such that*

$$\begin{cases} \sum_{i=0}^{\infty} 9^{-i} \phi(3^i x, 0) \text{ converges for all } x \in X, \text{and} \\ \lim_{n\to\infty} 9^{-n} \phi(3^n x, 3^n y) = 0 \text{ for all } x, y \in X, \end{cases} \tag{3.1}$$

or

$$\begin{cases} \sum_{i=1}^{\infty} 9^{i} \phi(3^{-i} x, 0) \text{ converges for all } x \in X, \text{and} \\ \lim_{n\to\infty} 9^{n} \phi(3^{-n} x, 3^{-n} y) = 0 \text{ for all } x, y \in X. \end{cases} \tag{3.2}$$

If a function $f : X \to Y$ satisfies the inequality

$$\|f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-16f(x)\| \le \phi(x, y) \quad \forall x, y \in X, \tag{3.3}$$

then there exists a unique function $Q : X \to Y$ *such that* Q *satisfies* (2.1) *and*

$$\|f(x) - Q(x)\| \le \begin{cases} \frac{1}{18}\sum_{i=0}^{\infty} 9^{-i}\phi(3^i x, 0) & \text{if (3.1) holds} \\ \frac{1}{18}\sum_{i=1}^{\infty} 9^{i}\phi(3^{-i} x, 0) & \text{if (3.2) holds} \end{cases} \quad \forall x \in X. \tag{3.4}$$

The function Q *is given by*

$$Q(x) = \begin{cases} \lim_{n\to\infty} 9^{-n} f(3^n x) & \text{if (3.1) holds} \\ \lim_{n\to\infty} 9^{n} f(3^{-n} x) & \text{if (3.2) holds} \end{cases} \tag{3.5}$$

Proof. We will first prove the case when the condition (3.1) holds. Letting $(x, y) = (x, 0)$ in (3.3), we get

$$\|2f(3x) - 18f(x)\| \le \phi(x, 0).$$

Dividing by 18, we have

$$\left\|\frac{f(3x)}{9} - f(x)\right\| \le \frac{1}{18}\phi(x, 0) \quad \forall x \in X. \tag{3.6}$$

Therefore,

$$\begin{aligned} \left\|\frac{f(3^n x)}{9^n} - f(x)\right\| &= \left\|\sum_{i=0}^{n-1}\left(\frac{f(3^{i+1}x)}{9^{i+1}} - \frac{f(3^i x)}{9^i}\right)\right\| \\ &\le \sum_{i=0}^{n-1}\frac{1}{9^i}\left\|\frac{f(3^{i+1}x)}{9} - f(3^i x)\right\| \\ &\le \frac{1}{18}\sum_{i=0}^{n-1}\frac{\phi(3^i x, 0)}{9^i} \end{aligned}$$

for a positive integer n and for all $x \in X$.

We have to show that the sequence $\left\{\frac{f(3^n x)}{9^n}\right\}$ converges for all $x \in X$. For every positive integer n and m, consider

$$\begin{aligned} \left\|\frac{f(3^{n+m}x)}{9^{n+m}} - \frac{f(3^n x)}{9^n}\right\| &= \frac{1}{9^n}\left\|\frac{f(3^m \cdot 3^n x)}{9^m} - f(3^n x)\right\| \\ &\le \frac{1}{18\cdot 9^n}\sum_{i=0}^{m-1}\frac{\phi(3^i \cdot 3^n x, 0)}{9^i} \\ &\le \frac{1}{18}\sum_{i=0}^{\infty}\frac{\phi(3^{i+n}x, 0)}{9^{i+n}} \end{aligned}$$

By condition (3.1), the right-hand side approaches 0 when n tends to infinity. Thus, the sequence is a Cauchy sequence. Since a Banach space is complete, we let

$$Q(x) = \lim_{n\to\infty}\frac{f(3^n x)}{9^n}$$

for all $x \in X$. By the definition of Q, we can see that inequality (3.4) holds in this case which the condition (3.1) holds.

To show that Q really satisfies the function equation, we set $(x, y) = (3^n x, 3^n y)$ in (3.3) and divide by 9^n, yielding

$$\frac{1}{9^n} \left\| f\left(3^n(3x+y)\right) + f\left(3^n(3x-y)\right) - f\left(3^n(x+y)\right) \right. \\ \left. - f\left(3^n(x-y)\right) - 16f\left(3^n x\right) \right\| \leq \frac{\phi\left(3^n x, 3^n y\right)}{9^n}.$$

Take the limit as n goes to infinity and note the definition of Q, the above equation becomes

$$\left\| Q(3x+y) + Q(3x-y) - Q(x+y) - Q(x-y) - 16Q(x) \right\| \leq 0.$$

for all $x, y \in X$. Therefore, Q satisfies (2.1).

Finally, we prove the uniqueness of Q. Suppose that there exists another quadratic function $S : X \to Y$ such that S satisfies the functional equation (2.1) and

$$\|S(x) - f(x)\| \leq \frac{1}{18} \sum_{i=0}^{\infty} \frac{\phi\left(3^i x, 0\right)}{9^i}$$

for all $x \in X$. By Theorem 2.1, every solution of (2.1) is also a solution of

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$$

and it is straightforward to show that every solution has the quadratic property; i.e.,$Q(nx) = n^2 Q(x)$ for every positive integer n and for all $x \in X$. Therefore,

$$\begin{aligned} \|S(x) - Q(x)\| &= \frac{1}{9^n} \left\| S\left(3^n x\right) - Q\left(3^n x\right) \right\| \\ &\leq \frac{1}{9^n} \left\| S\left(3^n x\right) - f\left(3^n x\right) \right\| + \frac{1}{9^n} \left\| f\left(3^n x\right) - Q\left(3^n x\right) \right\| \\ &\leq 2 \cdot \frac{1}{9^n} \cdot \frac{1}{18} \sum_{i=0}^{\infty} \frac{\phi\left(3^i \cdot 3^n x, 0\right)}{9^i} \\ &\leq \frac{1}{9} \sum_{i=0}^{\infty} \frac{\phi\left(3^{i+n} x, 0\right)}{9^{i+n}} \end{aligned}$$

for all $x \in X$. By condition (3.1), the right-hand side goes to 0 as n tends to infinity, and it follows that $Q(x) = S(x)$ for all $x \in X$. Hence, Q is unique.

For the case that the condition (3.2) holds, we can state the proof in a similar manner as in the case which the condition (3.1) holds. Starting by setting $(x, y) = (\frac{x}{3}, 0)$ in (3.3) and dividing by 2, we get

$$\left\| f(x) - 9f\left(\frac{x}{3}\right) \right\| \leq \frac{1}{2} \phi\left(\frac{x}{3}, 0\right) \quad \forall x \in X. \tag{3.7}$$

and this equation can be extended to

$$\left\| f(x) - 9^n f\left(\frac{x}{3^n}\right) \right\| \le \frac{1}{18} \sum_{i=1}^{n} 9^i \phi(3^{-i}x, 0)$$

for a positive integer n and for all $x \in X$.
We can show that a sequence $\left\{9^n f\left(\frac{x}{3^n}\right)\right\}$ converges for all $x \in X$. After that we let,

$$Q(x) = \lim_{n\to\infty} 9^n f(3^{-n}x)$$

for all $x \in X$. We will omit the proof and the other proof can be produced accordingly.
Then the proof is complete. □

Corollary 3.2. *Let X be a real vector space and let Y be a Banach space. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(3x+y) + f(3x-y) - f(x+y) - f(x-y) - 16f(x)\| \le \varepsilon \quad \forall x, y \in X \tag{3.8}$$

for some real number $\varepsilon > 0$, then there exists a unique function $Q : X \to Y$ such that Q satisfies (2.1) and

$$\|f(x) - Q(x)\| \le \frac{\varepsilon}{16} \quad \forall x \in X. \tag{3.9}$$

Proof. We choose $\phi(x, y) = \varepsilon$ for all $x, y \in X$. Being in accordance with (3.1) in Theorem 3.1, it follows that

$$\|f(x) - Q(x)\| \le \frac{1}{18} \sum_{i=0}^{\infty} \frac{\varepsilon}{9^i} = \frac{\varepsilon}{16}$$

for all $x \in X$ as desired. □

Corollary 3.3. *Let X be a normed vector space and let Y be a Banach space. Given positive real number ε and p with $p \neq 2$. If a function $f : X \to Y$ satisfies the inequality*

$$\|f(3x+y) + f(3x-y) - f(x+y) - f(x-y) - 16f(x)\| \le \varepsilon\big(\|x\|^p + \|y\|^p\big) \tag{3.10}$$

for all $x, y \in X$, then there exists a unique function $Q : X \to Y$ such that Q satisfies (2.1) and

$$\|f(x) - Q(x)\| \le \frac{\varepsilon}{2|9 - 3^p|} \|x\|^p \quad \forall x \in X. \tag{3.11}$$

Proof. We choose $\phi(x,y) = \varepsilon\left(\|x\|^p + \|y\|^p\right)$ for all $x, y \in X$.
If $0 < p < 2$, then the condition (3.1) in Theorem 3.1 is fulfilled, and consequently

$$\begin{aligned} \|f(x) - Q(x)\| &\leq \frac{1}{18}\sum_{i=0}^{\infty} \frac{\varepsilon\left(3^{ip}\|x\|^p\right)}{9^i} \\ &= \frac{1}{18}\sum_{i=0}^{\infty} \varepsilon \cdot 3^{i(p-2)}\|x\|^p \\ &= \frac{\varepsilon}{18} \cdot \frac{1}{1-3^{p-2}}\|x\|^p \\ &= \frac{\varepsilon\|x\|^p}{2\left(9-3^p\right)} \end{aligned}$$

for all $x \in X$.
If $p > 2$, the condition (3.2) in Theorem 3.1 is fulfilled, and consequently

$$\begin{aligned} \|f(x) - Q(x)\| &\leq \frac{1}{18}\sum_{i=1}^{\infty} 9^i \cdot \varepsilon \cdot \frac{\|x\|^p}{3^{ip}} \\ &= \frac{1}{18}\sum_{i=1}^{\infty} \frac{\varepsilon\|x\|^p}{3^{i(p-2)}} \\ &= \frac{\varepsilon}{18} \cdot \frac{\|x\|^p}{3^{p-2}-1} \\ &= \frac{\varepsilon\|x\|^p}{2\left(3^p-9\right)} \end{aligned}$$

for all $x \in X$.
Both cases of consideration complete our proof. □

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