Approximation Method for Fixed Points of Nonlinear Mapping and Variational Inequalities with Application

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Abstract: In this paper, we introduce the new method of iterative scheme \( \{x_n\} \) for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without demiclose condition and \( T_\omega := (1 - \omega)I + \omega T \), when \( T \) is a quasi-nonexpansive mapping and \( \omega \in (0, \frac{1}{2}) \) in a framework of Hilbert space. Using our main result, we obtain strong convergence theorems involving a finite family of nonspreading mapping and another corollary.

Keywords: quasi-nonexpansive mapping; variational inequality; fixed point; nonspreading mapping.

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1 Introduction

Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). We denote \( F(T) \) by the set of all fixed points of \( T \). Recall that the mapping \( T : C \to C \) is

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said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and
\[
\|Tx - p\| \leq \|x - p\|,
\]
for all $x \in C$ and $p \in F(T)$. Fixed point problems have been investigated in the following literature; see [1–3].

A mapping $A : C \to H$ is called $\alpha$-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that
\[
\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2,
\]
for all $x, y \in C$.

Let $B : C \to H$. The variational inequality is to find a point $u \in C$ such that
\[
\langle Bu, v - u \rangle \geq 0,
\]
for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C, B)$.

The variational inequalities were initially studied and introduced by Stampacchia [4, 5]. This problem is widely used in economics, social sciences and other fields, see for example [6–8].

Let $D_1, D_2 : C \to H$ be two mappings. In 2008, Ceng et al. [9] introduced a problem for finding $(x^\ast, z^\ast) \in C \times C$ such that
\[
\begin{cases}
\langle \lambda_1 D_1 z^\ast + x^\ast - z^\ast, x - x^\ast \rangle \geq 0, \forall x \in C, \\
\langle \lambda_2 D_2 x^\ast + z^\ast - x^\ast, x - z^\ast \rangle \geq 0, \forall x \in C,
\end{cases}
\]
which is called a system of variational inequalities where $\lambda_1, \lambda_2 > 0$.

In 2013, Kangtunyakarn [10] modified (1.2) for finding $(x^\ast, z^\ast) \in C \times C$ such that
\[
\begin{cases}
\langle x^\ast - (I - \lambda_1 D_1)(ax^\ast + (1 - a)z^\ast), x - x^\ast \rangle \geq 0, \forall x \in C, \\
\langle z^\ast - (I - \lambda_2 D_2)x^\ast, x - z^\ast \rangle \geq 0, \forall x \in C,
\end{cases}
\]
which is called a modification of system of variational inequalities, for every $\lambda_1, \lambda_2 > 0$ and $a \in [0,1]$. If $a = 0$, (1.3) reduces to (1.2). He introduced the relation between solutions of (1.3) and fixed point of the mapping $G$ as follows:

**Lemma 1.1.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $D_1, D_2 : C \to H$ be mappings. For every $\lambda_1, \lambda_2 > 0$ and $a \in [0,1]$, the following statements are equivalent:

1. $(x^\ast, z^\ast) \in C \times C$ is a solution of problem (1.3),
2. $x^\ast$ is a fixed point of the mapping $G : C \to C$, i.e., $x^\ast \in F(G)$, defined by

\[
G(x) = PC(I - \lambda_1 D_1)(ax + (1 - a)PC(I - \lambda_2 D_2)x),
\]

where $z^\ast = PC(I - \lambda_2 D_2)x^\ast$. 
Moreover, he introduced a new iterative algorithm \( \{x_n\} \) for finding a common element of the set of fixed points of a finite family of \( \kappa_i \)-strictly pseudo-contractive mappings and the set of solutions of problem (1.3) in Hilbert space. The sequence \( \{x_n\} \) is generated by

\[
\begin{align*}
  y_n &= P_C(I - \lambda_2 D_2)x_n, \\
  x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n SPC (ax_n + (1 - a)y_n - \lambda_1 D_1 (ax_n + (1 - a)y_n)), \forall n \geq 1,
\end{align*}
\]

where \( D_1, D_2 : C \to H \) are \( d_1, d_2 \)-inverse strongly monotone mappings, respectively, and \( S : C \to C \) is S-mapping generated by a finite family of strictly pseudo-contractive mapping and finite real numbers. Under suitable conditions of the parameters \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \lambda_1, \lambda_2, a, \) he proved a strong convergence theorem of iterative scheme \( \{x_n\} \).

In 2012, Tian and Jin [11] proved the following strong convergence theorem of iterative scheme \( \{x_n\} \) generated by (1.4).

**Theorem 1.2.** Starting with an arbitrary chosen \( x_1 \in H \), let the sequence \( \{x_n\} \) be generated by

\[
x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)T_\omega x_n, \tag{1.4}
\]

where the sequence \( \{\alpha_n\} \subset (0,1) \) satisfies \( \lim_{n \to \infty} \alpha_n = 0 \), and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Also \( \omega \in (0, \frac{1}{2}), T_\omega := (1 - \omega)I + \omega T \) with two conditions on \( T \):

1. \( \|Tx - q\| \leq \|x - q\| \) for any \( x \in H \), and \( q \in F(T) \); this means that \( T \) is a quasi-nonexpansive mapping;
2. \( T \) is demiclosed on \( H \); that is: if \( \{y_k\} \subset H, y_k \rightharpoonup z, \) and \( (I - T)y_k \to 0, \) then \( z \in F(T) \).

Then \( \{x_n\} \) converges strongly to the \( x^* \in F(T) \) which is the unique solution of the VIP:

\[
\langle (\gamma f - A)x^*, x - x^* \rangle \leq 0, \forall x \in F(T).
\]

Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping \( T \) by assuming the following conditions:

1. \( T_\omega := (1 - \omega)I + \omega T \),
2. \( T \) is demiclosed on \( H \).

see for example [12] and [13].

Motivated by [10], we introduced the new method for finding a common element of the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of a modified system of variational inequalities without the conditions (1) and (2) in a framework of Hilbert space. Using our main result, we obtain strong convergence theorems involving a finite family of nonspreading mapping and another corollary.
2 Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Throughout this paper, we denote weak and strong convergence by notations $\overset{w}{\rightharpoonup}$ and $\overset{w}{\rightarrow}$, respectively. For every $x \in H$, there exists a unique nearest point $P_C x$ in $C$ such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. $P_C$ is called the metric projection of $H$ onto $C$.

**Remark 2.1.** It is well-known that metric projection $P_C$ has the following properties:

1. $P_C$ is firmly nonexpansive, i.e.,
   \[ \|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \forall x, y \in H. \]

2. For each $x \in H$,
   \[ z = P_C (x) \iff \langle x - z, z - y \rangle \geq 0, \forall y \in C. \]

Recall that $H$ satisfies Opial’s condition [14], i.e., for any sequence $\{x_n\}$ with $x_n \overset{w}{\rightharpoonup} x$, the inequality
\[ \liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\| \]
holds for every $y \in H$ with $y \neq x$.

**Lemma 2.2.** Let $H$ be a real Hilbert space. Then there holds the following well-known results:

1. $\|x \pm y\|^2 = \|x\|^2 \pm 2 \langle x, y \rangle + \|y\|^2$,
2. $\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, x + y \rangle$,

for all $x, y \in H$.

**Lemma 2.3 ([15]).** Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have
\[
\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha \beta \|x - y\|^2 - \alpha \gamma \|x - z\|^2 - \beta \gamma \|y - z\|^2.
\]

**Lemma 2.4 ([16]).** Let $E$ be a uniformly convex Banach space, let $C$ be a nonempty closed convex subset of $E$ and let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.

**Lemma 2.5 ([17]).** Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying
\[ s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \forall n \geq 1 \]
where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that
(1) \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

(2) \( \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \) or \( \sum_{n=1}^{\infty} |\delta_n| < \infty \).

Then, \( \lim_{n \to \infty} s_n = 0 \).

**Lemma 2.6** ([10]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( D_1, D_2 : C \to H \) be \( d_1, d_2 \)-inverse strongly monotone mappings, respectively, which \( \text{VI} (C, D_1) \cap \text{VI} (C, D_2) \neq \emptyset \). Define a mapping \( G : C \to C \) by

\[
G (x) = P_C (I - \lambda_1 D_1) (ax + (1-a) P_C (I - \lambda_2 D_2) x),
\]

for every \( \lambda_1 \in (0, 2d_1), \lambda_2 \in (0, 2d_2) \) and \( a \in (0, 1) \). Then \( F (G) = \text{VI} (C, D_1) \cap \text{VI} (C, D_2) \).

**Lemma 2.7** ([15]). Let \( H \) be a real Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \) and let \( A \) be a mapping of \( C \) into \( H \). Let \( u \in C \). Then for \( \lambda > 0 \),

\[
\| u - \| P_C (I - \lambda A) u \| \leq u \in \text{VI} (C, A),
\]

where \( P_C \) is the metric projection of \( H \) onto \( C \).

The next result is very important for our main result.

**Lemma 2.8.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( T : C \to C \) be a quasi-nonexpansive mapping. Then \( \text{VI} (C, I - T) = F (T) \).

**Proof.** It is easy to see that \( F (T) \subseteq \text{VI} (C, I - T) \).

Let \( u \in \text{VI} (C, I - T) \), then we have

\[
\langle v - u, (I - T) u \rangle \geq 0, \ \forall v \in C.
\] (2.1)

Let \( v^* \in F (T) \), then we have

\[
\| Tu - v^* \|^2 \leq \| u - v^* \|^2.
\] (2.2)

On the other hand

\[
\| Tu - v^* \|^2 = \| (u - v^*) - (I - T) u \|^2
= \| u - v^* \|^2 - 2 (u - v^*, (I - T) u) + \| (I - T) u \|^2.
\] (2.3)

From (2.2) and (2.3), we have

\[
\| u - v^* \|^2 - 2 (u - v^*, (I - T) u) + \| (I - T) u \|^2 \leq \| u - v^* \|^2.
\]

From (2.1), we have

\[
\| (I - T) u \|^2 \leq 2 (u - v^*, (I - T) u).
\]

It follows that \( u \in F (T) \). Hence \( \text{VI} (C, I - T) \subseteq F (T) \). \( \Box \)

**Remark 2.9.** From Lemma 2.7 and 2.8 we have

\[
F (T) = \text{VI} (C, I - T) = F (P_C (I - \lambda (I - T))),
\]

for all \( \lambda > 0 \).
3 Main Results

Theorem 3.1. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and let $T : C \to C$ be a quasi-nonexpansive mapping. Let $A, B : C \to H$ be $\alpha, \beta$-inverse strongly monotone mappings, respectively. Define the mapping $G : C \to C$ by $Gx = P_C(I - \lambda_1 A)(ax + (1 - a)P_C(I - \lambda_2 B)x)$ for all $x \in C$. Assume $F = VI(C, A) \cap VI(C, B) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n Gx_n, \ \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ for all $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_F u$.

Proof. We divide the proof into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

Let $x, y \in C$. Since $A$ is $\alpha$-inverse strongly monotone and $\lambda_1 \in (0, 2\alpha)$, we have

$$\| (I - \lambda_1 A)x - (I - \lambda_1 A)y \|^2 = \| x - y \|^2 - 2\lambda_1 \langle x - y, Ax - Ay \rangle + \lambda_1^2 \| Ax - Ay \|^2$$
$$\leq \| x - y \|^2 - 2\alpha \lambda_1 \| Ax - Ay \|^2 + \lambda_1^2 \| Ax - Ay \|^2$$
$$= \| x - y \|^2 + \lambda_1 (1 - 2\alpha) \| Ax - Ay \|^2$$
$$\leq \| x - y \|^2.$$

Therefore $(I - \lambda_1 A)$ is a nonexpansive mapping. Similarly, $(I - \lambda_2 B)$ is a nonexpansive mapping. Hence $P_C(I - \lambda_1 A)$ and $P_C(I - \lambda_2 B)$ are nonexpansive mappings. From definition of the mapping $G$, we have $G$ is a nonexpansive mapping.

Let $x^* \in F$. From Remark 2.9, we have

$$x^* \in F(P_C(I - \lambda_n(I - T))).$$

By Lemma 2.6 we have

$$x^* = G(x^*) = P_C(I - \lambda_1 A)(ax^* + (1 - a)P_C(I - \lambda_2 B)x^*).$$
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Observe that

\[
\|T x_n - T x^*\|^2 = \|(x_n - x^*) - (I - T)x_n\|^2 = \|x_n - x^*\|^2 - 2\langle x_n - x^*, (I - T)x_n \rangle + \|(I - T)x_n\|^2.
\]

Since \(T\) is a quasi-nonexpansive mapping, we have

\[
\|(I - T)x_n\|^2 \leq 2\langle x_n - x^*, (I - T)x_n \rangle. \quad (3.2)
\]

From the nonexpansiveness of \(P_C\) and (3.2), we have

\[
\|P_C(I - \lambda_n(I - T)) x_n - x^*\|^2 = \|P_C(I - \lambda_n(I - T)) x_n - P_C(I - \lambda_n(I - T)) x^*\|^2 \\
\leq \|(I - \lambda_n(I - T)) x_n - (I - \lambda_n(I - T)) x^*\|^2 \\
= \||(x_n - x^*) - \lambda_n((I - T)x_n - (I - T)x^*))\|^2 \\
= \|x_n - x^*\|^2 - 2\lambda_n\langle x_n - x^*, (I - T)x_n \rangle \\
+ \lambda_n^2 \|(I - T)x_n\|^2 \\
\leq \|x_n - x^*\|^2 - \lambda_n\|(I - T)x_n\|^2 + \lambda_n^2 \|(I - T)x_n\|^2 \\
\leq \|x_n - x^*\|^2. \quad (3.3)
\]

Put \(M_n = ax_n + (1 - a)P_C(I - \lambda_2 B)x_n\) and \(W_n = P_C(I - \lambda_1 A)M_n\). From (3.1), we have

\[
x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n W_n.
\]

From the definition of \(x_n\), (3.3) and nonexpansiveness of \(G\), we have

\[
\|x_{n+1} - x^*\| = \|\alpha_n (u - x^*) + \beta_n (P_C(I - \lambda_n(I - T)) x_n - x^*) + \gamma_n (W_n - x^*)\| \\
\leq \alpha_n \|u - x^*\| + \beta_n \|P_C(I - \lambda_n(I - T)) x_n - x^*\| + \gamma_n \|W_n - x^*\| \\
\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| \\
+ \gamma_n \|P_C(I - \lambda_1 A)(ax_n + (1 - a)P_C(I - \lambda_2 B)x_n) - P_C(I - \lambda_1 A)(ax^* + (1 - a)P_C(I - \lambda_2 B)x^*)\| \\
= \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|G(x_n) - G(x^*)\| \\
\leq \alpha_n \|u - x^*\| + \beta_n \|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\
= \alpha_n \|u - x^*\| + (1 - \alpha_n) \|x_n - x^*\| \\
\leq \max\{\|u - x^*\|, \|x_n - x^*\|\}.
\]

By induction, we can conclude that

\[
\|x_n - x^*\| \leq \max\{\|u - x^*\|, \|x_1 - x^*\|\},
\]

for all \(n \geq 1\). This implies that the sequence \(\{x_n\}\) is bounded and so is \(\{(I - T)x_n\}\).
**Step 2.** We show that \( \lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \)

From the definition of \( x_n \) and nonexpansiveness of \( G \), we have

\[
\| x_{n+1} - x_n \| = \|(\alpha_n - \alpha_{n-1})u + (\beta_n - \beta_{n-1})P_C(I - \lambda_{n-1}(I - T))x_{n-1}
+ \beta_n(P_C(I - \lambda_n(I - T))x_n - P_C(I - \lambda_{n-1}(I - T))x_{n-1})
+ \gamma_n(W_n - W_{n-1}) + (\gamma_n - \gamma_{n-1})W_{n-1}\|
\leq |\alpha_n - \alpha_{n-1}| \| u \| + |\beta_n - \beta_{n-1}| \| P_C(I - \lambda_{n-1}(I - T))x_{n-1}\|
+ |\beta_n| \| P_C(I - \lambda_n(I - T))x_n - P_C(I - \lambda_{n-1}(I - T))x_{n-1}\|
+ |\gamma_n| \| W_n - W_{n-1}\| + |\gamma_n - \gamma_{n-1}| \| W_{n-1}\|
\leq (1 - \alpha_n) \| x_n - x_{n-1}\| + \lambda_n \| (I - T)x_n - (I - T)x_{n-1}\|
+ |\alpha_n - \alpha_{n-1}| \| u \| + |\beta_n - \beta_{n-1}| \| P_C(I - \lambda_{n-1}(I - T))x_{n-1}\|
+ |\gamma_n - \gamma_{n-1}| \| W_{n-1}\| + |\lambda_n - \lambda_{n-1}| \| (I - T)x_{n-1}\|
\leq (1 - \alpha_n) \| x_n - x_{n-1}\| + \lambda_n M + |\alpha_n - \alpha_{n-1}| M + |\beta_n - \beta_{n-1}| M
+ |\gamma_n - \gamma_{n-1}| M + |\lambda_n - \lambda_{n-1}| M,
\]

where \( M := \max_{n \in \mathbb{N}} \{ \|(I - T)x_{n+1} - (I - T)x_n\|, \|u\|, \|P_C(I - \lambda_n(I - T))x_n\|, \|W_n\|, \|(I - T)x_n\| \}. \)

From the condition (ii), (iv), (v) and Lemma 2.5, we have

\[
\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \tag{3.4}
\]

**Step 3.** We show that \( \lim_{n \to \infty} \| P_C(I - \lambda_n(I - T))x_n - x_n \| = 0. \)

Since \( x^* = P_C(I - A_1A) (ax^* + (1 - a)P_C(I - \lambda_2B)x^*) \) and \( M^* = ax^* + (1 - a)P_C(I - \lambda_2B)x^* \), we have \( x^* = P_C(I - \lambda_1A)M^*. \)
Since \( x^* \in VI(C, B) \), we obtain

\[
M^* - x^* = (1 - a) \left( P_C(I - \lambda_2 B)x^* - x^* \right) \\
= (1 - a) \left( P_C(I - \lambda_2 B)x^* - P_C(I - \lambda_2 B)x^* \right) \\
= 0. 
\]  

(3.5)

From the definition of \( M_n \) and \( M^* \), we have

\[
\|M_n - M^*\| = \|a(x_n - x^*) + (1 - a) \left( P_C(I - \lambda_2 B)x_n - P_C(I - \lambda_2 B)x^* \right)\| \\
\leq a\|x_n - x^*\| + (1 - a) \|P_C(I - \lambda_2 B)x_n - P_C(I - \lambda_2 B)x^*\| \\
\leq a\|x_n - x^*\| + (1 - a)\|x_n - x^*\| \\
= \|x_n - x^*\|. 
\]  

(3.6)

From the definition of \( W_n \), we have

\[
\|W_n - x^*\|^2 = \|P_C(I - \lambda_1 A)M_n - P_C(I - \lambda_1 A)M^*\|^2 \\
\leq (\|I - \lambda_1 A\| M_n - (I - \lambda_1 A)M^*, W_n - x^*) \\
= \frac{1}{2} \left( \|I - \lambda_1 A\| M_n - (I - \lambda_1 A)M^*\|^2 + \|W_n - x^*\|^2 \\
- \|I - \lambda_1 A\| M_n - (I - \lambda_1 A)M^* - W_n + x^*\|^2 \right) \\
\leq \frac{1}{2} \left( \|M_n - M^*\|^2 + \|W_n - x^*\|^2 \\
- \|(M_n - W_n) - \lambda_1 (AM_n - AM^*)\|^2 \right),
\]

which implies that

\[
\|W_n - x^*\|^2 \leq \|M_n - M^*\|^2 - \|(M_n - W_n) - \lambda_1 (AM_n - AM^*)\|^2 \\
= \|M_n - M^*\|^2 - \|M_n - W_n\|^2 + 2\lambda_1 \langle M_n - W_n, AM_n - AM^* \rangle \\
- \lambda_1^2 \|AM_n - AM^*\|^2. 
\]  

(3.7)

From the definition of \( W_n \), we have

\[
\|W_n - x^*\|^2 \\
= \|P_C(I - \lambda_1 A)M_n - P_C(I - \lambda_1 A)M^*\|^2 \\
\leq (\|I - \lambda_1 A\| M_n - (I - \lambda_1 A)M^*, W_n - x^*) \\
= \|M_n - M^*\|^2 - \lambda_1 (AM_n - AM^*)\|^2 \\
= \|M_n - M^*\|^2 - 2\lambda_1 \langle M_n - M^*, AM_n - AM^* \rangle + \lambda_1^2 \|AM_n - AM^*\|^2 \\
\leq \|M_n - M^*\|^2 - 2\lambda_1 \alpha \|AM_n - AM^*\|^2 + \lambda_1^2 \|AM_n - AM^*\|^2 \\
= \|M_n - M^*\|^2 - \lambda_1 (2\alpha - \lambda_1) \|AM_n - AM^*\|^2. 
\]  

(3.8)
From the definition of $x_n$, (3.3), (3.6) and (3.8), we have

$$
\|x_{n+1} - x^*\| \leq \alpha_n \|u - x^*\|^2 + \beta_n \|PC(I - \lambda_n(I - T))x_n - x^*\|^2
+ \gamma_n \|W_n - x^*\|^2
\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2
+ \gamma_n \left(\|M_n - M^*\|^2 - \lambda_1(2\alpha - \lambda_1) \|AM_n - AM^*\|^2\right)
\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2
- \gamma_n \lambda_1(2\alpha - \lambda_1) \|AM_n - AM^*\|^2
= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2
- \gamma_n \lambda_1(2\alpha - \lambda_1) \|AM_n - AM^*\|^2.
$$

It implies that

$$
\gamma_n \lambda_1(2\alpha - \lambda_1) \|AM_n - AM^*\|^2
\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2
\leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\| \left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right).
$$

From the condition $(ii)$ and (3.4), we derive

$$
\lim_{n \to \infty} \|AM_n - AM^*\| = 0.
$$

(3.10)

From the definition of $x_n$, (3.3), (3.6) and (3.7), we have

$$
\|x_{n+1} - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + \beta_n \|PC(I - \lambda_n(I - T))x_n - x^*\|^2 + \gamma_n \|W_n - x^*\|^2
\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2
+ \gamma_n \left(\|M_n - M^*\|^2 - \|M_n - W_n\|^2 + 2\lambda_1 \|M_n - W_n, AM_n - AM^*\|
- \lambda_1^2 \|AM_n - AM^*\|^2\right)
\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \|x_n - x^*\|^2 - \gamma_n \|M_n - W_n\|^2
+ 2\lambda_1 \|M_n - W_n\| \|AM_n - AM^*\|
= \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 - \gamma_n \|M_n - W_n\|^2
+ 2\lambda_1 \|M_n - W_n\| \|AM_n - AM^*\|.
$$

It follows that

$$
\gamma_n \|M_n - W_n\|^2 \leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2
+ 2\lambda_1 \|M_n - W_n\| \|AM_n - AM^*\|
\leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\| \left(\|x_n - x^*\| + \|x_{n+1} - x^*\|\right)
+ 2\lambda_1 \|M_n - W_n\| \|AM_n - AM^*\|.
$$

(3.11)
From the condition \( (ii) \), \( \text{(3.8)} \) and \( \text{(3.10)} \), we derive

\[
\lim_{n \to \infty} \| M_n - W_n \| = 0. \tag{3.12}
\]

From the property of \( P_C \), we have

\[
\| P_C(I - \lambda_2 B)x_n - x^* \|^2 = \| P_C(I - \lambda_2 B)x_n - P_C(I - \lambda_2 B)x^* \|^2 \\
\leq \langle (I - \lambda_2 B)x_n - (I - \lambda_2 B)x^* , P_C(I - \lambda_2 B)x_n - x^* \rangle \\
= \frac{1}{2} \left( \| (I - \lambda_2 B)x_n - x^* \|^2 \\
+ \| P_C(I - \lambda_2 B)x_n - x^* \|^2 \\
- \| (I - \lambda_2 B)x_n - (I - \lambda_2 B)x^* - P_C(I - \lambda_2 B)x_n + x^* \|^2 \right) \\
\leq \frac{1}{2} \left( \| x_n - x^* \|^2 + \| P_C(I - \lambda_2 B)x_n - x^* \|^2 \\
- \| (x_n - P_C(I - \lambda_2 B)x_n) - \lambda_2 (Bx_n - Bx^*) \|^2 \right). \tag{3.13}
\]

This implies that

\[
\| P_C(I - \lambda_2 B)x_n - x^* \|^2 \leq \| x_n - x^* \|^2 \\
- \| (x_n - P_C(I - \lambda_2 B)x_n) - \lambda_2 (Bx_n - Bx^*) \|^2 \\
= \| x_n - x^* \|^2 - \| x_n - P_C(I - \lambda_2 B)x_n \|^2 \\
+ 2\lambda_2 \langle x_n - P_C(I - \lambda_2 B)x_n, Bx_n - Bx^* \rangle \\
- \lambda_2^2 \| Bx_n - Bx^* \|^2. \tag{3.14}
\]

By using the same method as \( \text{(3.9)} \), we have

\[
\| P_C(I - \lambda_2 B)x_n - x^* \|^2 \leq \| x_n - x^* \|^2 - \lambda_2 (2\beta - \lambda_2) \| Bx_n - Bx^* \|^2. \tag{3.15}
\]

Since \( x^* \in VI(C, A) \), we have

\[
\| W_n - x^* \|^2 = \| P_C(I - \lambda_1 A)M_n - P_C(I - \lambda_1 A)x^* \|^2 \\
\leq \| ax_n + (1 - a)P_C(I - \lambda_2 B)x_n - x^* \|^2 \\
= \| a(x_n - x^*) + (1 - a)(P_C(I - \lambda_2 B)x_n - x^*) \|^2 \\
\leq a \| x_n - x^* \|^2 + (1 - a) \| P_C(I - \lambda_2 B)x_n - x^* \|^2. \tag{3.16}
\]

From the definition of \( x_n \), \( \text{(3.3)} \), \( \text{(3.4)} \) and \( \text{(3.15)} \), we have

\[
\| x_{n+1} - x^* \|^2 \\
\leq \alpha_n \| u - x^* \|^2 + \beta_n \| P_C(I - \lambda_n(I - T))x_n - x^* \|^2 + \gamma_n \| W_n - x^* \|^2 \\
\leq \alpha_n \| u - x^* \|^2 + \beta_n \| P_C(I - \lambda_n(I - T))x_n - x^* \|^2 \\
+ \gamma_n \left( a \| x_n - x^* \|^2 + (1 - a) \| P_C(I - \lambda_2 B)x_n - x^* \|^2 \right).
\]
This implies that
\[
\lim_{n \to \infty} \|Bx_n - Bx^*\| = 0. \tag{3.17}
\]
From the condition \((ii)\) and \((3.4)\), we have
\[
\|x_{n+1} - x^*\|^2 \leq \alpha_n \|u - x^*\|^2 + \beta_n \|P_C(I - \lambda_n(I - T))x_n - x^*\|^2
+ \gamma_n \left( a \|x_n - x^*\|^2 + (1 - a) \|P_C(I - \lambda_2 B)x_n - x^*\|^2 \right)
\leq \alpha_n \|u - x^*\|^2 + \beta_n \|x_n - x^*\|^2 + \gamma_n \left( a \|x_n - x^*\|^2 + (1 - a) \|x_n - x^*\|^2
+ (1 - a)(\|x_n - x^*\|^2 - \|x_n - P_C(I - \lambda_2 B)x_n\|^2
+ 2\lambda_2 \|x_n - P_C(I - \lambda_2 B)x_n, Bx_n - Bx^*\) - \lambda_2^2 \|Bx_n - Bx^*\|^2) \right)
\leq \alpha_n \|u - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2
- \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\|^2
+ 2\lambda_2 \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n, Bx_n - Bx^*\)
\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\|^2
+ 2\lambda_2 \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\| \|Bx_n - Bx^*\|. \tag{3.18}
\]
This implies that
\[
\gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\|^2
\leq \alpha_n \|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2
+ 2\lambda_2 \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\| \|Bx_n - Bx^*\|
\leq \alpha_n \|u - x^*\|^2 + \|x_n - x_{n+1}\| \|x_n - x^*\| + \|x_{n+1} - x^*\|)
+ 2\lambda_2 \gamma_n(1 - a) \|x_n - P_C(I - \lambda_2 B)x_n\| \|Bx_n - Bx^*\|. \tag{3.18}
\]
From the condition \((ii)\), \((3.4)\) and \((3.17)\), we derive
\[
\lim_{n \to \infty} \|x_n - P_C(I - \lambda_2 B)x_n\| = 0.
\]
Since
\[
\|M_n - x_n\| = \|ax_n + (1 - a)P_C(I - \lambda_2 B)x_n - x_n\|
= (1 - a)\|P_C(I - \lambda_2 B)x_n - x_n\|
\]
and \(\|P_C(I - \lambda_2 B)x_n - x_n\| \to 0\) as \(n \to \infty\), we have
\[
\lim_{n \to \infty} \|M_n - x_n\| = 0. \tag{3.19}
\]
From (3.12) and (3.19), we have
\[
\lim_{n \to \infty} \|W_n - x_n\| = 0. \tag{3.20}
\]
Since
\[
x_{n+1} - x_n = \alpha_n(u - x_n) + \beta_n(P_C(I - \lambda_n(I - T))x_n - x_n) + \gamma_n(W_n - x_n),
\]
it implies by the condition (ii), the condition (iii), (3.3) and (3.20) that
\[
\lim_{n \to \infty} \|P_C(I - \lambda_n(I - T))x_n - x_n\| = 0. \tag{3.21}
\]
**Step 4.** We show that \(\limsup_{n \to \infty} \langle u - z_0, x_n - z_0 \rangle \leq 0\), where \(z_0 = P_F u\). To show this inequality, take a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) such that
\[
\limsup_{n \to \infty} \langle u - z_0, x_n - z_0 \rangle = \lim_{j \to \infty} \langle u - z_0, x_{n_j} - z_0 \rangle.
\]
Without loss of generality, we may assume that \(x_{n_j} \rightharpoonup \omega\) as \(j \to \infty\), where \(\omega \in C\). First, we show that \(\omega \in F(T)\). From Remark 2.9, we have \(F(T) = VI(C, I - T) = F(P_C(I - \lambda_n(I - T)))\). Assume that \(\omega \notin F(T)\), that \(\omega \neq P_C(I - \lambda_n(I - T))\omega\).
By \(x_{n_j} \rightharpoonup \omega\) as \(j \to \infty\), (3.21) and Opial’s property, we have
\[
\liminf_{j \to \infty} \|x_{n_j} - \omega\| < \liminf_{j \to \infty} \|x_{n_j} - P_C(I - \lambda_{n_j}(I - T))\omega\|
\leq \liminf_{j \to \infty} \|x_{n_j} - P_C(I - \lambda_{n_j}(I - T))x_{n_j}\|
+ \|P_C(I - \lambda_{n_j}(I - T))x_{n_j} - P_C(I - \lambda_{n_j}(I - T))\omega\|
\leq \liminf_{j \to \infty} \|x_{n_j} - P_C(I - \lambda_{n_j}(I - T))x_{n_j}\|
+ \|x_{n_j} - \omega\| + \lambda_{n_j} \|(I - T)x_{n_j} - (I - T)\omega\|
\leq \liminf_{j \to \infty} \|x_{n_j} - \omega\|.
\]
This is a contradiction, we have
\[
\omega \in F(T). \tag{3.22}
\]
Next, we show that \( \omega \in VI(C, A) \cap VI(C, B) \). From Lemma 2.4, we have \( VI(C, A) \cap VI(C, B) = F(G) \). From (3.22), we have \( W_n \to \omega \) as \( j \to \infty \).

\[
\|W_n - G(W_n)\| = \|P_C(I - \lambda_n A)(ax_n + (1 - a)P_C(I - \lambda_2 B)x_n) - G(W_n)\|
\]
\[
= \|G(x_n) - G(W_n)\|
\]
\[
\leq \|x_n - W_n\|.
\]

From (3.21), we have
\[
\lim_{n \to \infty} \|W_n - G(W_n)\| = 0.
\]

From \( W_{n_j} \to \omega \) as \( j \to \infty \) and Lemma 2.4, we have
\[
\omega \in F(G) = VI(C, A) \cap VI(C, B).
\]

From (3.22) and (3.23), we have \( \omega \in \mathcal{F} \). Since \( x_{n_j} \to \omega \) as \( j \to \infty \), we have
\[
\lim_{n \to \infty} \sup(u - z_0, x_n - z_0) = \lim_{j \to \infty} (u - z_0, x_{n_j} - z_0)
\]
\[
= \langle u - z_0, \omega - z_0 \rangle \leq 0.
\]

**Step 5.** Finally, we show that the sequence \( \{x_n\} \) converges strongly to \( z_0 = P_{\mathcal{F}}u \). From the definition of \( x_n \) and \( z_0 = P_{\mathcal{F}}u \), we have
\[
\|x_{n+1} - z_0\|^2 = \|\alpha_n(u - z_0) + \beta_n(P_C(I - \lambda_n(I - T))x_n - z_0) + \gamma_n(W_n - z_0)\|^2
\]
\[
\leq \|\beta_n(P_C(I - \lambda_n(I - T))x_n - z_0) + \gamma_n(W_n - z_0)\|^2
\]
\[
+ 2\alpha_n\langle u - z_0, x_{n+1} - z_0 \rangle
\]
\[
\leq \beta_n \|P_C(I - \lambda_n(I - T))x_n - z_0\|^2 + \gamma_n \|W_n - z_0\|^2
\]
\[
+ 2\alpha_n\langle u - z_0, x_{n+1} - z_0 \rangle
\]
\[
\leq \beta_n \|x_n - z_0\|^2 + \gamma_n \|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0 \rangle
\]
\[
= (1 - \alpha_n) \|x_n - z_0\|^2 + 2\alpha_n\langle u - z_0, x_{n+1} - z_0 \rangle.
\]

From the condition (ii), (3.21) and Lemma 2.5, we can conclude that the sequence \( \{x_n\} \) converges strongly to \( z_0 = P_{\mathcal{F}}u \). This completes the proof. \( \square \)

From our main result, Lemma 1.1 and Lemma 2.6, we have the following corollary:

**Corollary 3.2.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( T : C \to C \) be a quasi-nonexpansive mapping. Let \( A, B : C \to H \) be \( \alpha \), \( \beta \)-inverse strongly monotone mappings, respectively. Define the mapping \( G : C \to C \) by \( Gx = P_C(I - \lambda_1 A)(ax + (1 - a)P_C(I - \lambda_2 B)x) \) for all \( x \in C \). Assume \( \mathcal{F} = F(G) \cap F(T) \neq \emptyset \). Suppose that \( x_1, u \in C \) and let \( \{x_n\} \) be a sequence generated by
\[
x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + r_n Gx_n, \quad \forall n \geq 1,
\]
where \( \lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta) \) and \( \{\alpha_n\}, \{\beta_n\}, \{r_n\} \) are sequences in \([0, 1] \). Suppose the following conditions holds:
(i) $\alpha_n + \beta_n + \gamma_n = 1$,  
(ii) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,  
(iii) $0 < a \leq \beta_n \leq c < 1$ for all $n \geq 1$,  
(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,  
(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_{\text{F}(u)}$ and $(z_0, y_0)$ is a solution of (1.3), where $y_0 = P_{C}(I - \lambda_2 B)z_0$.

4 Application

In this section, we prove strong convergence theorems involving the set of fixed points of nonspreading mapping.

A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \forall x, y \in C.$$  

The such mapping is defined by Kohsaka and Takahashi [19].

The following lemma is needed to prove in application.

Lemma 4.1 ([19]). Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, and let $S$ be a nonspreading mapping of $C$ into itself. Then $F(S)$ is closed and convex.

In 2009, Kangtunyakarn and Suantai [20] introduced the $S$-mapping generated by $T_1, T_2, ..., T_N$ and $\lambda_1, \lambda_2, ..., \lambda_N$ as following. Let $C$ be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^{N}$ be a finite family of (nonexpansive) mappings of $C$ into itself. For each $j = 1, 2, ..., N$, let $\alpha_j = (\alpha_{1j}, \alpha_{2j}, \alpha_{3j}) \in I \times I \times I$, where $I \in [0, 1]$ and $\alpha_{1j} + \alpha_{2j} + \alpha_{3j} = 1$. Define the mapping $S : C \to C$ as follows:

$U_0 = I,$  
$U_1 = \alpha_{11}T_1U_0 + \alpha_{12}U_0 + \alpha_{13}I,$  
$U_2 = \alpha_{11}^2T_2U_1 + \alpha_{12}^2U_1 + \alpha_{13}^2I,$  
$U_3 = \alpha_{11}^3T_3U_2 + \alpha_{12}^3U_2 + \alpha_{13}^3I,$  
\[ \vdots \]  
$U_{N-1} = \alpha_{11}^{N-1}T_{N-1}U_{N-2} + \alpha_{12}^{N-1}U_{N-2} + \alpha_{13}^{N-1}I,$  
$S = U_N = \alpha_{11}^N T_N U_{N-1} + \alpha_{12}^N U_{N-1} + \alpha_{13}^N I.$
This mapping is called an \( S \)-mapping generated by \( T_1, T_2, ..., T_N \) and \( \alpha_1, \alpha_2, ..., \alpha_N \).

For every \( i = 1, 2, ..., N \). Put \( \alpha_i^3 = 0 \) in Definition 4.1, then the \( S \)-mapping is reduced to the \( K \)-mapping defined by Kangtunyakarn and Suantai \([21]\) as following. Let \( C \) be a nonempty convex subset of a real Banach space. Let \( \{ T_i \}_{i=1}^N \) be a finite family of mappings of \( C \) into itself, and let \( \lambda_1, \lambda_2, ..., \lambda_N \) be real numbers such that \( 0 \leq \lambda_i \leq 1 \) for every \( i = 1, 2, ..., N \). We define a mapping \( K : C \rightarrow C \) as follows:

\[
U_0 = I, \\
U_1 = \lambda_1 T_1 + (1 - \lambda_1) I, \\
U_2 = \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\
U_3 = \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\
\vdots \\
U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\
K = U_N = \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}.
\]

Such a mapping \( K \) is called the \( K \)-mapping generated by \( T_1, T_2, ..., T_N \) and \( \lambda_1, \lambda_2, ..., \lambda_N \).

**Lemma 4.2** \((22)\). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \{ T_i \}_{i=1}^N \) be a finite family of nonsnapping mappings of \( C \) into \( C \) with \( \bigcap_{i=1}^N F(T_i) \neq \emptyset \), and let \( \alpha_3 = (\alpha_1^3, \alpha_2^3, \alpha_3^3) \in I \times I \times I, j = 1, 2, ..., N \), where \( I = [0, 1] \), \( \alpha_1^j + \alpha_2^j + \alpha_3^j = 1 \), \( \alpha_1^j, \alpha_3^j \in (0, 1) \) for all \( j = 1, 2, ..., N - 1 \) and \( \alpha_1^N \in (0, 1], \alpha_3^N \in [0, 1], \alpha_2^j \in (0, 1) \) for all \( j = 1, 2, ..., N \). Let \( S \) be the mapping generated by \( T_1, T_2, ..., T_N \) and \( \alpha_1, \alpha_2, ..., \alpha_N \). Then \( F(S) = \bigcap_{i=1}^N F(T_i) \) and \( S \) is a quasinonexpanse mapping.

**Lemma 4.3** \((23)\). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \{ T_i \}_{i=1}^N \) be a finite family of nonsnapping mappings of \( C \) into itself with \( \bigcap_{i=1}^N F(T_i) \neq \emptyset \) and let \( \lambda_1, \lambda_2, ..., \lambda_N \) be real numbers such that \( 0 < \lambda_i < 1 \) for every \( i = 1, 2, ..., N - 1 \) and \( 0 < \lambda_N \leq 1 \). Let \( K \) be the \( K \)-mapping generated by \( T_1, T_2, ..., T_N \) and \( \lambda_1, \lambda_2, ..., \lambda_N \). Then \( F(K) = \bigcap_{i=1}^N F(T_i) \) and \( K \) is quasinonexpanse mapping.

By using these results, we obtain the following theorems

**Theorem 4.4.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \{ T_i \}_{i=1}^N \) be a finite family of nonsnapping mappings of \( C \) into \( C \) with \( \bigcap_{i=1}^N F(T_i) \neq \emptyset \), and let \( \alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I, j = 1, 2, ..., N \), where \( I = [0, 1] \), \( \alpha_1^j + \alpha_2^j + \alpha_3^j = 1 \), \( \alpha_1^j, \alpha_3^j \in (0, 1) \) for all \( j = 1, 2, ..., N - 1 \) and \( \alpha_1^N \in (0, 1] \), \( \alpha_2^j \in (0, 1] \) for all \( j = 1, 2, ..., N \). Then \( \bigcap_{i=1}^N F(T_i) \neq \emptyset \) and \( F(K) = \bigcap_{i=1}^N F(T_i) \) and \( K \) is quasinonexpanse mapping.
Then \( 0 \), \( \sum_{j=1}^{N} \alpha_j^N \in [0, 1) \) for all \( j = 1, 2, \ldots, N \). Let \( S \) be the mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \alpha_1, \alpha_2, \ldots, \alpha_N \). Let \( A, B : C \to H \) be \( \alpha, \beta \)-inverse strongly monotone mappings, respectively. Define the mapping \( G : C \to C \) by \( Gx = PC(I - \lambda_1 A)(ax + (1 - a)PC(I - \lambda_2 B)x) \) for all \( x \in C \). Assume \( F = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). Suppose that \( x_1, u \in C \) and let \( \{x_n\} \) be sequence generated by
\[ x_{n+1} = \alpha_n u + \beta_n PC(I - \lambda_n (I - S))x_n + \gamma_n Gx_n, \forall n \geq 1, \]
where \( \lambda_n \in (0, 2\alpha), \lambda_2 \in (0, 2\beta) \) and \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \([0, 1)\). Suppose the following conditions hold:

(i) \( \alpha_n + \beta_n + \gamma_n = 1 \),

(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

(iii) \( 0 < a \leq \beta_n \leq c < 1 \) for all \( n \geq 1 \),

(iv) \( \sum_{n=1}^{\infty} \lambda_n < \infty \) and \( 0 < \lambda_n < 1 \),

(v) \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty \).

Then \( \{x_n\} \) converges strongly to \( z_0 = P_F u \).

Proof. By using Theorem 3.1 and Lemma 4.2 we obtain the conclusion. \qed

**Theorem 4.5.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \{T_i\}_{i=1}^{N} \) be a finite family of nonspreading mappings of \( C \) into itself with \( \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \) and let \( \lambda_1, \lambda_2, \ldots, \lambda_N \) be real numbers such that \( 0 < \lambda_i < 1 \) for every \( i = 1, 2, \ldots, N-1 \) and \( 0 < \lambda_N \leq 1 \). Let \( K \) be the \( K \)-mapping generated by \( T_1, T_2, \ldots, T_N \) and \( \lambda_1, \lambda_2, \ldots, \lambda_N \). Let \( A, B : C \to H \) be \( \alpha, \beta \)-inverse strongly monotone mappings, respectively. Define the mapping \( G : C \to C \) by \( Gx = PC(I - \lambda_1 A)(ax + (1 - a)PC(I - \lambda_2 B)x) \) for all \( x \in C \). Assume \( F = VI(C, A) \cap VI(C, B) \cap \bigcap_{i=1}^{N} F(T_i) \neq \emptyset \). Suppose that \( x_1, u \in C \) and let \( \{x_n\} \) be sequence generated by
\[ x_{n+1} = \alpha_n u + \beta_n PC(I - \lambda_n (I - K))x_n + \gamma_n Gx_n, \forall n \geq 1, \]
where \( \lambda_1 \in (0, 2\alpha), \lambda_2 \in (0, 2\beta) \) and \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) are sequences in \([0, 1)\). Suppose the following conditions hold:

(i) \( \alpha_n + \beta_n + \gamma_n = 1 \),

(ii) \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \),

(iii) \( 0 < a \leq \beta_n \leq c < 1 \) for all \( n \geq 1 \),

(iv) \( \sum_{n=1}^{\infty} \lambda_n < \infty \) and \( 0 < \lambda_n < 1 \),

(v) \( \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty \).

Then \( \{x_n\} \) converges strongly to \( z_0 = P_F u \).
Proof. By using Theorem 3.1 and Lemma 4.3, we obtain the conclusion.

The following result is direct proved from Theorem 4.4. Therefore, we omit the prove.

**Corollary 4.6.** Let $C$ be a nonempty closed convex subset of a real Hilbert space. Let $T$ be a nonspreading mappings of $C$ into itself with $F(T) \neq \emptyset$. Let $A, B: C \to H$ be $\alpha, \beta$-inverse strongly monotone mappings, respectively. Define the mapping $G: C \to C$ by

$$Gx = P_C(I - \lambda_1 A)(ax + (1 - a)P_C(I - \lambda_2 B)x)$$

for all $x \in C$. Assume $F = VI(C, A) \cap VI(C, B) \cap F(T) \neq \emptyset$. Suppose that $x_1, u \in C$ and let $\{x_n\}$ be sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n P_C(I - \lambda_n(I - T))x_n + \gamma_n Gx_n, \forall n \geq 1,$$

where $\lambda_1 \in (0, 2\alpha)$, $\lambda_2 \in (0, 2\beta)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$. Suppose the following conditions hold:

(i) $\alpha_n + \beta_n + \gamma_n = 1$,

(ii) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(iii) $0 < a \leq \beta_n \leq c < 1$ for all $n \geq 1$,

(iv) $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $0 < \lambda_n < 1$,

(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $z_0 = P_Fu$.

**References**


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