Existence of Positive Solutions to a Second-order Multi-Point Boundary Value Problem with Delay

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In this paper, by using the Krasnosel’skii fixed-point theorem, we study the existence of positive solutions to the second-order delay differential equation,

\[ u''(t) + \lambda a(t)f(t, u(t - \tau)) = 0, \quad t \in J = [0, 1], \]
\[ u(t) = \beta u(\eta), \quad -\tau \leq t \leq 0, \]
\[ u(1) = \alpha u(\eta), \]

where \( \alpha, \beta, \eta \) are constants with \( \eta \in (0, 1) \). \( \lambda \) is a positive real parameter.

**Keywords**: Positive solution; Fixed point; Multi-point boundary value problem; Delay.

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1 Introduction

Multi-point boundary value problems for ordinary differential equations arise in a variety of different areas of applied mathematics and physics. The study of the three-point boundary value problem for nonlinear ordinary differential equations was initiated by Gupta [1-2]. Since then, nonlinear multi-point boundary value problems have been studied by several authors. For details, see, for example, [3-7] and reference therein.

In this paper, we consider the existence of positive solutions for the following multi-point boundary value problem of the second order delay
differential equation:
\begin{align*}
  u''(t) + \lambda a(t)f(t, u(t - \tau)) &= 0, \quad t \in J = [0, 1], \\
  u(t) &= \beta u(\eta), \quad -\tau \leq t \leq 0, \\
  u(1) &= \alpha u(\eta),
\end{align*}

where \( \alpha, \beta, \eta \) are constants with \( \eta \in (0, 1) \). \( \lambda \) is a positive real parameter.

For the case \( 0 < \tau < \frac{1}{2} \), \( \beta = \alpha = 0 \), Bai and Xu [6] studied the existence of multiple positive solutions to BVP (1.1) with \( a(t)f(t, u) = g(t, u) \) by using Krasnoselskii fixed-point theorem.

Recently, for the case \( 0 < \tau < 1 \), \( \beta = 0 \), Wang and Shen [7] given some sufficient conditions with \( \lambda \) belonging to an open interval of eigenvalues to ensure the existence of positive solutions to BVP (1.1).

2 Preliminaries

In this section we give the following definition of positive solution of (1.1).

**Definition 2.1.** \( u(t) \) is called a positive solution of (1.1) if \( u \in C[-\tau, 1] \cap C^2(0, 1) \), \( u(t) > 0 \) for \( t \in (0, 1) \) and satisfies (1.1).

**Lemma 2.1.** Assume \( \beta \neq \frac{1-\alpha\eta}{1-\eta} \). Then for \( y \in C([0, 1], \mathbb{R}) \), the problem
\begin{align*}
  u''(t) + y(t) &= 0, \quad 0 < t < 1, \\
  u(0) &= \beta u(\eta), \\
  u(1) &= \alpha u(\eta), \quad (2.1, 2.2)
\end{align*}

has a unique solution
\begin{align*}
  u(t) &= \int_0^1 G(t, s)y(s)ds + \frac{\beta + (\alpha - \beta)t}{(1-\alpha\eta) - \beta(1-\eta)} \int_0^1 G(\eta, s)y(s)ds, \quad (2.3)
\end{align*}

where
\begin{align*}
  G(t, s) &= \begin{cases} 
  s(1-t), & 0 \leq s \leq t \leq 1, \\
  t(1-s), & 0 \leq t < s \leq 1.
\end{cases} \quad (2.4)
\end{align*}

**Proof.** It is well known that the Green’s function is \( G(t, s) \) as in (2.4) for the second-order two point linear boundary value problem
\begin{align*}
  \begin{cases}
    w'' + y(t) = 0, & 0 < t < 1 \\
    w(0) = 0, \\
    w(1) = 0,
  \end{cases} \quad (2.5)
\end{align*}
and the solution of (2.5) is given by
\[ w(t) = \int_0^1 G(t, s)y(s)ds, \]
and
\[ w(0) = 0, \quad w(1) = 0, \quad w(\eta) = \int_0^1 G(\eta, s)y(s)ds. \]  \hspace{1cm} (2.6)
We suppose that the solution of the three-point boundary value problem (2.1), (2.2) can be expressed by
\[ u(t) = w(t) + A + Bt, \]  \hspace{1cm} (2.7)
where \( A \) and \( B \) are constants that will be determined.

From (2.6), (2.7) we know that
\[
\begin{align*}
u(0) &= A, \\
u(1) &= A + B, \\
u(\eta) &= w(\eta) + A + B\eta. 
\end{align*}
\]
Putting these into (2.2) yields
\[
\begin{align*}
(1 - \beta)A - \beta\eta B &= \beta w(\eta), \\
(1 - \alpha)A + (1 - \alpha\eta)B &= \alpha w(\eta).
\end{align*}
\]
Since \( \beta \neq \frac{1 - \alpha\eta}{1 - \eta} \), solving the system of linear equations on the unknowns \( A \), \( B \), we obtain
\[
\begin{align*}
A &= \frac{\beta w(\eta)}{(1 - \alpha\eta) - \beta(1 - \eta)}, \\
B &= \frac{(\alpha - \beta)w(\eta)}{(1 - \alpha\eta) - \beta(1 - \eta)}.
\end{align*}
\]
Hence
\[ u(t) = w(t) + \frac{\beta + (\alpha - \beta)t}{(1 - \alpha\eta) - \beta(1 - \eta)}w(\eta). \]
This implies that
\[ u(t) = \int_0^1 G(t, s)y(s)ds + \frac{\beta + (\alpha - \beta)t}{(1 - \alpha\eta) - \beta(1 - \eta)} \int_0^1 G(\eta, s)y(s)ds. \]
Next, we will show that the solution \( u(t) \) is unique. Assume that \( v(t) \) is another solution of the three-point boundary value problem (2.1), (2.2).
Let $z(t) = v(t) - u(t)$, $t \in [0, 1]$. Then, we get $z''(t) = v''(t) - u''(t) = 0$, $t \in [0, 1]$ therefore

$$z(t) = C_1 t + C_2,$$  \hspace{1cm} (2.8)

where $C_1$, $C_2$ are undetermined constants. From (2.2), we have

$$z(0) = \beta z(\eta), \quad z(1) = \alpha z(\eta).$$  \hspace{1cm} (2.9)

Using (2.8), we obtain

$$z(0) = C_2, \quad z(1) = C_1 + C_2, \quad z(\eta) = C_1 \eta + C_2.$$  \hspace{1cm} (2.10)

From (2.9), (2.10) we know that

$$\begin{align*}
-\beta \eta C_1 + (1 - \beta) C_2 &= 0, \\
(1 - \alpha \eta) C_1 + (1 - \alpha) C_2 &= 0.
\end{align*}$$

Since $\beta \neq \frac{1 - \alpha \eta}{1 - \eta}$, then the system of linear equations on the unknown numbers $C_1$, $C_2$, has exactly one solution, therefore $z(t) \equiv 0$, $t \in [0, 1]$, so $v(t) = u(t)$, $t \in [0, 1]$, that is uniqueness of the solution.

**Lemma 2.2.** [5] Let $0 < \alpha < \frac{1}{\eta}$, $0 < \beta < \frac{1 - \alpha \eta}{1 - \eta}$. If $f \in C([0, 1], [0, \infty))$, then the unique solution to problem (2.1), (2.2) satisfies

$$u(t) \geq 0, \quad t \in [0, 1].$$

**Lemma 2.3.** [5] Let $\alpha \eta \neq 1$, $\beta > \max\{\frac{1 - \alpha \eta}{1 - \eta}, 0\}$ and $f \in C([0, 1], [0, \infty))$, then problem (2.1), (2.2) has no nonnegative solutions.

Hence, in this paper, we always assume the following condition is satisfied

(i): $0 < \eta < 1$, $0 < \alpha < \frac{1}{\eta}$, $0 < \beta < \frac{1 - \alpha \eta}{1 - \eta}$, $0 < \tau < 1$, $a : (0, 1) \rightarrow [0, \infty)$ is continuous and $f : [0, 1] \times [0, \infty) \rightarrow [0, \infty)$ is continuous, $0 < \int_0^1 s(1 - s)a(s)ds < \infty$. There exist constants $0 \leq b < c \leq 1 - \tau$ such that

$$\int_b^{c + \tau} a(s)ds > 0.$$

By Lemma 2.1-2.3, it is easy to see that the BVP (1.1) has a solution $u = u(t)$ if and only if $u$ is a solution of the operator equation $u = Tu$, where

$$Tu(t) = \begin{cases}
\beta u(\eta), & -\tau \leq t \leq 0, \\
\lambda \int_0^1 G(t, s)a(s)f(s, u(s - \tau))ds \\
\quad + \lambda \frac{\beta + (\alpha - \beta)\tau}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(\eta, s)a(s)f(s, u(s - \tau))ds, & 0 \leq t < s \leq 1.
\end{cases}$$  \hspace{1cm} (2.11)
Let

\[ P = \{ u \in C[-\tau, 1] : u(t) \geq 0 \text{ for } t \in (-\tau, 1], u(t) = \beta u(\eta), -\tau \leq t \leq 0, \quad u(1) = \alpha u(\eta) \}. \]

It is clear that \( C[-\tau, 1] \) with norm \( \|u\| = \sup \{|u(t)| : -\tau \leq t \leq 1\} \) is a Banach space.

Put

\[ p(t) = \min \left\{ \frac{\beta \eta + (1 - \beta) t}{\eta}, \frac{1 - \alpha \eta + (\alpha - 1) t}{1 - \eta} \right\}, \]

\[ \theta = \min \left\{ \frac{\beta \eta + (1 - \beta) b}{\eta}, \frac{1 - \alpha \eta + (\alpha - 1) c}{1 - \eta} \right\} \min \{\eta, 1 - \eta\}, \]

and a cone \( K \) in \( C[-\tau, 1] \) is defined by

\[ K = \left\{ u \in C[-\tau, 1] : u(t) \geq 0 \text{ for } t \in J, \min_{b \leq t \leq c} u(t) \geq \theta \|u\| \right\}. \]

We now state and prove the following lemmas before stating our main results.

**Lemma 2.4.** Assume that \( u \in P \cap C^2(0, 1) \) and \( u''(t) \leq 0 \) for \( t \in J \), then \( u(t) \geq p(t) \min\{\eta, 1 - \eta\}\|u\| \) for \( t \in J \).

**Proof.** Firstly, we show that \( u(\eta) \geq \min\{\eta, 1 - \eta\}\|u\| \). By the properties of Green's function (2.4), we can find that

\[ \min\{\eta, 1 - \eta\} s(1-s) \leq G(\eta, s) \leq G(s, s) = s(1-s), \quad (\eta, s) \in [0, 1] \times [0, 1]. \]

(2.12)

By using (2.11) and (2.12), we know that for every solution \( u(t) \) of BVP (1.1), one has

\[ \|u\| \leq \lambda \int_0^1 s(1-s)a(s)f(s, u(s-\tau))ds \]

\[ + \frac{\lambda \alpha}{(1-\alpha \eta) - \beta(1-\eta)} \int_0^1 G(\eta, s)a(s)f(s, u(s-\tau))ds. \]

(2.13)
Multiplying both side of inequality (2.13) by \( \min\{\eta, 1 - \eta\} \), we get

\[
\min\{\eta, 1 - \eta\} \|u\| \leq \lambda \int_0^1 \min\{\eta, 1 - \eta\} s(1-s)a(s)f(s, u(s-\tau))ds \\
+ \frac{\lambda \alpha \min\{\eta, 1 - \eta\}}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(\eta, s)a(s)f(s, u(s-\tau))ds \\
\leq \lambda \int_0^1 G(\eta, s)a(s)f(s, u(s-\tau))ds \\
+ \frac{\lambda(\beta + (\alpha - \beta)\eta)}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(\eta, s)a(s)f(s, u(s-\tau))ds = u(\eta).
\]

Hence, the inequality \( u(\eta) \geq \min\{\eta, 1 - \eta\} \|u\| \) is true.

Next, we will prove that \( u(t) \geq p(t) \min\{\eta, 1 - \eta\} \|u\| \) in two cases.

Case (i) if \( 0 \leq t \leq \eta \), then

\[
\frac{u(t) - u(0)}{t - 0} \geq \frac{u(\eta) - u(0)}{\eta - 0}.
\]

Using \( u(0) = \beta u(\eta) \), we have

\[
u(t) \geq \frac{\beta \eta + (1 - \beta)t}{\eta} u(\eta) \geq \frac{\beta \eta + (1 - \beta)t}{\eta} \min\{\eta, 1 - \eta\} \|u\|.
\]

Case (ii) if \( \eta < t \leq 1 \), then

\[
\frac{u(t) - u(\eta)}{t - \eta} \geq \frac{u(1) - u(\eta)}{1 - \eta},
\]

since \( u(1) = \alpha u(\eta) \), we get

\[
u(t) \geq \frac{1 - \alpha \eta + (\alpha - 1)t}{1 - \eta} u(\eta) \geq \frac{1 - \alpha \eta + (\alpha - 1)t}{1 - \eta} \min\{\eta, 1 - \eta\} \|u\|.
\]

Combining above two cases, we have that \( u(t) \geq p(t) \min\{\eta, 1 - \eta\} \|u\| \) for \( t \in J \), and the proof is complete.

**Lemma 2.5.** The fixed point of \( T \) is a solution of (1.1) and \( T : K \to K \) is completely continuous.

**Proof.** From (2.11), we have

\[
(Tu)''(t) + \lambda a(t)f(t, u(t-\tau)) = 0 \quad t \in J = [0, 1], \\
(Tu)(t) = \beta(Tu)(\eta), \quad -\tau \leq t \leq 0, \\
(Tu)(1) = \alpha(Tu)(\eta).
\]
Therefore, the fixed point of $T$ is a solution of (1.1).

Next, we will prove that $T : K \to K$. For any $u \in K$, it is easy to see that $Tu \in C[-\tau, 1]$ and $Tu \geq 0$ for $t \in J$. Since $(Tu)''(t) \leq 0$, by Lemma 2.4, we have

$$(Tu)(t) \geq p(t) \min\{\eta, 1-\eta\} ||Tu||,$$

for $t \in J$.

Hence

$$\min\{(Tu)(t) : b \leq t \leq c\} \geq \min\{p(t) : b \leq t \leq c\} \min\{\eta, 1-\eta\} ||Tu|| \geq \theta ||Tu||.$$

So $T : K \to K$. By using Arzela-Ascoli theorem, it is easy to prove that $T$ is completely continuous. The proof is complete. □

**Lemma 2.6.** Let $X$ be a Banach space, and let $K \subset X$ be a cone. Assume $\Omega_1, \Omega_2$ are open subset of $E$ with $0 \in \Omega_1, \overline{\Omega}_2 \subset \Omega_2$, and let

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K,$$

be a completely continuous operator such that either

(i) $||Au|| \leq ||u||, u \in K \cap \partial \Omega_1$ and $||Au|| \geq ||u||, u \in K \cap \partial \Omega_2$, or

(ii) $||Au|| \geq ||u||, u \in K \cap \partial \Omega_1$ and $||Au|| \leq ||u||, u \in K \cap \partial \Omega_2$.

Then $A$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

### 3 Main Results

Let

$$f^0 = \limsup_{u \to 0^+} \max_{t \in J} \frac{f(t, u)}{u}, \quad f_0 = \liminf_{u \to 0^+} \min_{t \in J} \frac{f(t, u)}{u},$$

$$f^\infty = \limsup_{u \to \infty} \max_{t \in J} \frac{f(t, u)}{u}, \quad f_\infty = \liminf_{u \to \infty} \min_{t \in J} \frac{f(t, u)}{u}.$$

**Theorem 3.1.** Let (H) hold and $f_\infty > 0, f^0 < \infty$, then there exists at least one positive solution to (1.1) for

$$\lambda \in \left( \frac{1}{f^\infty \sup_{t \in J} \left( \beta \min\{\eta, 1-\eta\} \int_0^\tau G(t, s)a(s)ds + \theta \int_{b+\tau}^{c+\tau} G(t, s)a(s)ds \right)}, \right.$$

$$\left. \frac{1 - \alpha \eta - \beta(1-\eta)}{(1 + \alpha(1-\eta) + \beta \eta f^0 \left( \beta \int_0^\tau G(s, s)a(s)ds + \int_\tau^1 G(s, s)a(s)ds \right) \theta \int_{b+\tau}^{c+\tau} G(t, s)a(s)ds \right)} \right)^{3.1}.$$
Proof. By (3.1), there exists an $\varepsilon > 0$ such that

\[
\frac{1}{(f_\infty - \varepsilon) \sup_{t \in J} \left( \beta \min \{\eta, 1-\eta\} \int_0^T G(t,s)a(s)ds + \theta \int_{b+\tau}^{c+\tau} G(t,s)a(s)ds \right)} \leq \lambda \leq \frac{1 - \alpha \eta - \beta (1-\eta)}{(1 + \alpha (1-\eta) + \beta \eta)(f^0 + \varepsilon) \left( \int_0^1 G(s,s)a(s)ds + \int_0^1 G(s,s)a(s)ds \right)}.
\]

Let $\varepsilon$ be fixed. By $f^0 < \infty$, there exists a $r > 0$ such that for $u : 0 < u \leq r$,

\[
f(s, u) \leq (f^0 + \varepsilon)u.
\]

Let $\Omega_1 = \{ u \in C[-\tau,1] : \|u\| < r \}$, then for $u \in K \cap \partial \Omega_1$, we have by (3.2) and (3.3) that

\[
\|Tu\| \leq \lambda \int_0^1 G(s,s)a(s)f(s, u(s-\tau))ds \\
+ \frac{\lambda(\beta + \alpha)}{(1 - \alpha \eta) - \beta (1-\eta)} \int_0^1 G(s,s)a(s)f(s, u(s-\tau))ds \\
\leq \lambda \frac{1 + \alpha (1-\eta)}{(1 - \alpha \eta) - \beta (1-\eta)} \int_0^1 G(s,s)a(s)f(s, u(s-\tau))ds \\
\leq \lambda (f^0 + \varepsilon) \frac{1 + \alpha (1-\eta) + \beta \eta}{(1 - \alpha \eta) - \beta (1-\eta)} \int_0^1 G(s,s)a(s)u(s-\tau)ds \\
= \lambda (f^0 + \varepsilon) \frac{1 + \alpha (1-\eta) + \beta \eta}{(1 - \alpha \eta) - \beta (1-\eta)} \left( \int_0^\tau G(s,s)a(s)\beta u(\eta)ds \\
+ \int_\tau^1 G(s,s)a(s)u(s-\tau)ds \right) \\
\leq \lambda (f^0 + \varepsilon) \frac{1 + \alpha (1-\eta) + \beta \eta}{(1 - \alpha \eta) - \beta (1-\eta)} \left( \beta \int_0^\tau G(s,s)a(s)ds \\
+ \int_\tau^1 G(s,s)a(s)ds \right) \|u\| \leq \|u\|.
\]

Next, by $f_\infty > 0$, there exists a $R > r$ such that $f(s, u) \geq (f_\infty - \varepsilon)u$ for
\( u \geq R \). Set \( \Omega_2 = \{ u \in C[-\tau, 1] : \|u\| < R \} \), then for \( u \in K \cap \partial \Omega_2 \), we have

\[
\|Tu\| \geq \lambda \sup_{t \in J} \int_0^1 G(t,s) a(s) f(s, u(s - \tau)) \|u\| ds
\]
\[
\geq \lambda \sup_{t \in J} \int_0^1 G(t,s) a(s) (f_\infty - \varepsilon) u(s - \tau) \|u\| ds
\]
\[
= \lambda (f_\infty - \varepsilon) \sup_{t \in J} \left( \int_0^\tau G(t,s) a(s) \beta u(\eta) ds + \int_1^{1-\tau} G(t,s) a(s) u(s - \tau) ds \right)
\]
\[
= \lambda (f_\infty - \varepsilon) \sup_{t \in J} \left( \int_0^\tau G(t,s) a(s) \beta u(\eta) ds + \int_1^{1-\tau} G(t,s) a(s) u(s - \tau) ds \right)
\]
\[
\geq \lambda (f_\infty - \varepsilon) \sup_{t \in J} \left( \int_0^\tau G(t,s) a(s) \beta u(\eta) ds + \int_1^{1-\tau} G(t,s) a(s) u(s - \tau) ds \right)
\]
\[
\geq \lambda (f_\infty - \varepsilon) \sup_{t \in J} \left( \int_0^\tau G(t,s) a(s) \beta \min \{ \eta, 1 - \eta \} \|u\| ds
\]
\[
+ \int_0^{\tau} G(t,s) a(s) \theta \|u\| ds \right)
\]
\[
\geq \lambda (f_\infty - \varepsilon) \sup_{t \in J} \left( \beta \min \{ \eta, 1 - \eta \} \right) \int_0^\tau G(t,s) a(s) ds
\]
\[
+ \theta \int_0^{\tau} G(t,s) a(s) ds \right) \|u\| \geq \|u\|.
\]

Therefore, by the first part of Lemma 2.6, \( T \) has a fixed point \( u \in K \cap (\Omega_2 \setminus \Omega_1) \) and \( \|u\| \geq r \). From Lemma 2.5, \( u(t) \) is a positive solution of BVP (1.1). The proof is complete.

**Theorem 3.2.** Let (H) hold and \( f_0 > 0, \ f_\infty < \infty \), then there exists at least one positive solution to (1.1) for

\[
\lambda \in \left( \frac{1}{f_0 \sup_{t \in J} \left( \beta \min \{ \eta, 1 - \eta \} \int_0^\tau G(t,s) a(s) ds + \theta \int_0^{\tau} G(t,s) a(s) ds \right) + 1 - \alpha \eta - \beta (1 - \eta)}{(1 + \alpha (1 - \eta) + \beta \eta) f_\infty \left( \beta \int_0^\tau G(s,s) a(s) ds + \int_1^{1-\tau} G(s,s) a(s) ds \right)} \right).
\]

(3.4)
Proof. Suppose that $\lambda$ satisfies (3.4). There exists an $\varepsilon > 0$ such that

$$\frac{1}{(f_0 - \varepsilon) \sup_{t \in J} \left( \beta \min \{\eta, 1 - \eta\} \int_0^t G(t, s) a(s) ds \right)}$$

$$\leq \lambda \leq \frac{1 - \alpha \eta - \beta (1 - \eta)}{(1 + \alpha(1 - \eta) + \beta \eta)(f^\infty + \varepsilon) \left( \beta \int_0^t G(t, s) a(s) ds + \int_1^t G(t, s) a(s) ds \right)}$$

(3.5)

By $f_0 > 0$, there exists a $r^* > 0$ such that for $u : 0 < u \leq r^*$,

$$f(s, u) \geq (f_0 - \varepsilon) u.$$  (3.6)

Let $\Omega_1 = \{ x \in C[-\tau, 1] : \|u\| < r^* \}$, then for $u \in K \cap \partial \Omega_1$, we have by (3.5) and (3.6) that

$$\| Tu \| \geq \lambda \sup_{t \in J} \int_0^1 G(t, s) a(s) f(s, u(s - \tau)) ds$$

$$\geq \lambda \sup_{t \in J} \int_0^1 G(t, s) a(s)(f_0 - \varepsilon) u(s - \tau)) ds$$

$$= \lambda (f_0 - \varepsilon) \sup_{t \in J} \left( \int_0^t G(t, s) a(s) \beta u(\eta) ds + \int_0^1 G(t, s) a(s) u(s - \tau) ds \right)$$

$$= \lambda (f_0 - \varepsilon) \sup_{t \in J} \left( \int_0^t G(t, s) a(s) \beta u(\eta) ds + \int_0^{1-\tau} G(t, s + \tau) a(s + \tau) u(s) ds \right)$$

$$\geq \lambda (f_0 - \varepsilon) \sup_{t \in J} \left( \int_0^t G(t, s) a(s) \beta \min \{\eta, 1 - \eta\} \|u\| ds \right.$$

$$+ \int_b^c G(t, s + \tau) a(s + \tau) \theta \|u\| ds \left.) \right)$$

$$\geq \lambda (f_0 - \varepsilon) \sup_{t \in J} \left( \beta \min \{\eta, 1 - \eta\} \int_0^t G(t, s) a(s) ds \right.$$

$$+ \theta \int_b^{c+\tau} G(t, s) a(s) ds \) \|u\| \geq \|u\|.}
By $f^\infty < \infty$, we choose that $R_\ast > r^\ast$ such that for $u \geq R_\ast$,
\[ f(s,u) \leq (f^\infty + \varepsilon)u. \]

There are two cases of interest: Case (i) $f$ is bounded, and Case (ii) $f$ is unbounded.

Case (i) Suppose that $f$ is bounded. We can choose $N > r^\ast$ such that for $u \geq R^\ast$, $f(s,u) \leq N$ for $s \in J$ and $u \in [0,\infty)$. Let
\[ R^\ast = \max \left\{ N, \lambda N \frac{1 + \alpha(1 - \eta) + \beta \eta}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(s,s) a(s) ds \right\} \]
and $\Omega_2 = \{ x \in C[-\tau, 1] : \|u\| < R^\ast \}$. Then for $u \in K \cap \partial \Omega_2$, we have
\[
\|Tu\| \leq \lambda \int_0^1 G(s,s) a(s) f(s,u(s - \tau)) ds \\
+ \frac{\lambda(\beta + \alpha)}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(s,s) a(s) f(s,u(s - \tau)) ds \\
\leq \lambda N \frac{1 + \alpha(1 - \eta) + \beta \eta}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(s,s) a(s) ds \leq R^\ast = \|u\|.
\]

Case (ii) Suppose that $f$ is unbounded. There exists $R^{**} > R_\ast$ such that $f(s,u) \leq f(s,R^{**})$ for $s \in J$ and $0 < x \leq R^{**}$. Then for $u \in K \cap \partial \Omega_2$, we have
\[
\|Tu\| \leq \lambda \int_0^1 G(s,s) a(s) f(s,u(s - \tau)) ds \\
+ \frac{\lambda(\beta + \alpha)}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(s,s) a(s) f(s,u(s - \tau)) ds \\
\leq \lambda \frac{1 + \alpha(1 - \eta) + \beta \eta}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(s,s) a(s) f(s,R^{**}) ds \\
\leq \lambda (f^\infty + \varepsilon) R^{**} \frac{1 + \alpha(1 - \eta) + \beta \eta}{(1 - \alpha \eta) - \beta(1 - \eta)} \int_0^1 G(s,s) a(s) ds \leq R^{**} = \|u\|.
\]

Therefore, by the second part of Lemma 2.6, $T$ has a fixed point $u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$ and $\|u\| \geq r^\ast$. From Lemma 2.5, $u(t)$ is a positive solution of BVP (1.1). The proof is complete. \hfill $\square$

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References


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