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A Mixed-Type Quadratic and Cubic Functional Equation and Its Stability

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In this paper, we prove the general solution of a mixed-type quadratic and cubic functional equation

$$f(x+3y)-3f(x+2y)+3f(x+y)-f(x)=3f(y)-3f(-y)$$

and investigate its general stability.

Keywords : Functional Equation; Mixed-Type Quadratic and Cubic Functional Equation; Stability.
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1 Introduction

The stability problem was originated in 1940 by S.M. Ulam [5]. He proposed the following famous question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies the inequality

$$d\left(f(xy), f(x)f(y)\right) < \delta \quad \textit{for all } x, y \in G_1,$$

then there exists a homomorphism $H : G_1 \to G_2$ with

$$d\left(f(x), H(x)\right) < \varepsilon \quad \textit{for all } x \in G_1?$$

After that, in 1941, D.H. Hyers [3] published a theorem affirming an existence in the Ulam's problem for the case of approximately additive function $f : G_1 \to G_2$ where G_1 and G_2 are Banach spaces:

Assume that E_1 and E_2 are Banach spaces. If a function $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for some $\varepsilon \ge 0$ for all $x, y \in E_1$, then the limit

$$a(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in E_1$ and $a : E_1 \to E_2$ is the unique additive function such that

$$\|f(x) - a(x)\| \le \varepsilon$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then a is linear.

In 1950, T. Aoki [1] gave the generalized Hyers' theorem. Afterwards, in 1978, Th.M. Rassias [4] published the following stability theorem:

If a function $f : E_1 \to E_2$ between Banach spaces satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \theta\left(\|x\|^p + \|y\|^p\right)$$

for some $\theta \ge 0$ and $0 \le p < 1$ for all $x, y \in E_1$, then there exists an additive function $a : E_1 \to E_2$ such that

$$\|f(x) - a(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then a is linear.

This theorem stimulated a number of authors to investigate stability problems of various functional equations.

In this paper, we will determine the general solution of a mixed-type quadratic and cubic functional equation,

$$f(x+3y) - 3f(x+2y) + 3f(x+y) - f(x) = 3f(y) - 3f(-y),$$

and will also investigate its general stability.

2 Preliminaries

In this section, we will introduce generalized polynomial functions. For further details, please refer to the book authored by S. Czerwik [2].

Let X and Y be linear spaces over the field $\mathbb{Q}$ of rational numbers, and let $s = 0, 1, 2, \ldots$. A function $f : X \to Y$ is called a *polynomial function of order* s if f satisfies the functional equation

$$\sum_{i=0}^{s+1} (-1)^{s+1-i} \binom{s+1}{i} f(x + iy) = 0 \tag{2.1}$$

for all $x, y \in X$. For instance when $s = 1$, a function f fulfilling the functional equation

$$f(x + 2y) - 2f(x + y) + f(x) = 0 \tag{2.2}$$

is a polynomial function of order 1. The following theorem gives a formula of the general solution of the polynomial functions.

Theorem 2.1. *Let $n = 0, 1, 2, \ldots$. A function $f : X \to Y$ is a polynomial function of order n if and only if there exist k-additive symmetric functions $A_k : X^k \to Y, k = 0, 1, 2, \ldots, n$ such that*

$$f(x) = A^0(x) + A^1(x) + A^2(x) + \ldots + A^n(x)$$

for all $x \in X$ where $A^k : X \to Y, k = 0, 1, 2, \ldots, n$ is the diagonalization of A_k and is defined by

$$A^k(x) = A_k(\underbrace{x, ..., x}_{k}), \quad \textit{for all} \quad x \in X.$$

By the above theorem, a function f satisfying (2.2) take the form of $f(x) = A^0(x) + A^1(x)$. Let us consider a k-additive symmetric function $A_k(x_1, ..., x_k)$ for $x_1, x_2, \ldots, x_k \in X$ and its diagonalization, $A^k(x)$. It can be proven that the additivity of A_k in the i^{th} variable leads us to

$$A_k(x_1, \ldots, x_{i-1}, rx_i, x_{i+1}, \ldots, x_k) = rA_k(x_1, \ldots, x_k) \quad \text{for each} \quad r \in \mathbb{Q}.$$

Thus $A^k(rx) = r^k A^k(x)$. In particular, $A^k(-x) = (-1)^k A^k(x)$. Since the function A^1 satisfies the additive functional equation

$$A^1(x + y) = A^1(x) + A^1(y)$$

for all $x \in X$, $A^1(x)$ will also be called an *additive function*. In addition, the functional equation $A^2(x)$ and $A^3(x)$ will be referred a *quadratic function* and a *cubic function*, respectively.

In this paper, we will call a function $f : X \to Y$ given by

$$f(x) = A^0(x) + A^2(x) + A^3(x)$$

for all $x \in X$ a *mixed-type quadratic and cubic function*.

3 Main Results

3.1 The general solution

Theorem 3.1. *Let X and Y be vector spaces. A function $f : X \to Y$ satisfies the functional equation*

$$f(x+3y) - 3f(x+2y) + 3f(x+y) - f(x) = 3f(y) - 3f(-y), \tag{3.1}$$

for all $x, y \in X$ if and only if there exist a quadratic function $A^2 : X \to Y$, a cubic function $A^3 : X \to Y$ and a constant A^0 such that

$$f(x) = A^0 + A^2(x) + A^3(x) \tag{3.2}$$

for all $x \in X$.

Proof. Assume that a function $f : X \to Y$ satisfies (3.1). Replacing x by $x + y$ in (3.1) and taking the difference of the previous result and (3.1), we then obtain

$$f(x+4y) - 4f(x+3y) + 6f(x+2y) - 4f(x+y) + f(x) = 0. \tag{3.3}$$

Hence, by the Theorem 2.1 of the preliminaries section, f is a polynomial function of order 3 and take the form of

$$f(x) = A^0 + A^1(x) + A^2(x) + A^3(x) \tag{3.4}$$

for all $x \in X$. Substituting (3.4) into (3.1), one get that

$$6A^3(y) = 6A^1(y) + 6A^3(y).$$

Thus, it yields $A^1(y) = 0$ for all $y \in X$. □

3.2 The General Stability

In this section, the stability of the functional equation will be investigated. Define

$$Df(x,y) = f(x+3y) - 3f(x+2y) + 3f(x+y) - f(x) - 3f(y) + 3f(-y).$$

Theorem 3.2. *Let X be a real vector space, Y be a Banach space. Let $\phi : X^2 \to [0,\infty)$ be an even function with respect to each variable such that*

$$\begin{cases} \sum_{i=0}^{\infty} 2^{-i}\phi(2^i y, 2^i y) \text{ converges for all } y \in X, \text{and} \\ \lim_{s\to\infty} 2^{-s}\phi(2^s x, 2^s y) = 0 \text{ for all } x, y \in X, \end{cases} \tag{3.5}$$

or

$$\begin{cases} \sum_{i=1}^{\infty} 8^{i}\phi(2^{-i} y, 2^{-i} y) \text{ converges for all } y \in X, \text{and} \\ \lim_{s\to\infty} 8^{s}\phi(2^{-s} x, 2^{-s} y) = 0 \text{ for all } x, y \in X. \end{cases} \tag{3.6}$$

If a function $f : X \to Y$ satisfies the inequality

$$\|Df(x,y)\| \le \phi(x,y) \tag{3.7}$$

for all $x, y \in X$ and $f(0) = 0$ when (3.6) holds, then there exists a unique function $T : X \to Y$ that satisfies (3.1) and, for all $x \in X$,

$$\|f(x) - T(x)\| \le \begin{cases} \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\phi(2^i x, 2^i x) + \frac{1}{8}\sum_{i=0}^{\infty} 8^{-i}\phi(2^i x, 2^i x) + 2\|f(0)\| & \text{if (3.5) holds} \\ \frac{1}{4}\sum_{i=1}^{\infty} 4^{i}\phi(2^{-i} x, 2^{-i} x) + \frac{1}{8}\sum_{i=1}^{\infty} 8^{i}\phi(2^{-i} x, 2^{-i} x) & \text{if (3.6) holds.} \end{cases} \tag{3.8}$$

The function T is given by

$$T(x) = \begin{cases} \lim_{s\to\infty} 4^{-s} f_e(2^s x) + 8^{-s} f_o(2^s x) & \text{if (3.5) holds} \\ \lim_{s\to\infty} 4^{s} f_e(2^{-s} x) + 8^{s} f_o(2^{-s} x) & \text{if (3.6) holds} \end{cases}$$

for all $x \in X$.

Proof. Let F be a function on X defined by $F(x) = f(x) - f(0)$ for all $x \in X$. Then we have $F(0) = 0$. (3.7) can be rewritten as

$$\|F(x+3y) - 3F(x+2y) + 3F(x+y) - F(x) - 3F(y) + 3F(-y)\| \le \phi(x,y). \tag{3.9}$$

Putting $x = -y$ in (3.9), we get

$$\|F(2y) - 3F(y) + 3F(0) - F(-y) - 3F(y) + 3F(-y)\| \leq \phi(-y, y).$$

Simplifying the above equation yields

$$\|F(2y) - 6F(y) + 2F(-y)\| \leq \phi(y, y). \tag{3.10}$$

Reversing the sign of y in (3.10) and realising that ϕ is even, we have

$$\|F(-2y) - 6F(-y) + 2F(y)\| \leq \phi(y, y). \tag{3.11}$$

Define the even part and the odd part of function f by

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2},$$

respectively. We apply triangle inequality with (3.10) and (3.11) to obtain that

$$\|2F_e(2y) - 8F_e(y)\| \leq 2\phi(y, y)$$

and

$$\|2F_o(2y) - 16F_o(y)\| \leq 2\phi(y, y)$$

which is simplified to

$$\left\|F_e(y) - 4^{-1}F_e(2y)\right\| \leq \frac{1}{4}\phi(y, y) \tag{3.12}$$

and

$$\left\|F_o(y) - 8^{-1}F_o(2y)\right\| \leq \frac{1}{8}\phi(y, y). \tag{3.13}$$

For each positive integer s, we obtain

$$\begin{aligned}
\left\|F_e(y) - 4^{-s}F_e(2^s y)\right\| &= \left\|\sum_{i=0}^{s-1}\left(4^{-i}F_e(2^i y) - 4^{-(i+1)}F_e(2^{(i+1)}y)\right)\right\| \\
&\leq \sum_{i=0}^{s-1} 4^{-i}\left\|F_e(2^i y) - 4^{-1}F_e(2 \cdot 2^i y)\right\| \\
&\leq \frac{1}{4}\sum_{i=0}^{s-1} 4^{-i}\phi(2^i y, 2^i y). \qquad (3.14)
\end{aligned}$$

Similarly, for each positive integer s,

$$\left\|F_o(y) - 8^{-s}F_o(2^s y)\right\| \leq \frac{1}{8}\sum_{i=0}^{s-1} 8^{-i}\phi(2^i y, 2^i y).$$

In order to prove the convergence of the sequence $\{4^{-s}F_e(2^s y)\}_{s=1}^{\infty}$, we divide inequality (3.14) by 4^{-t} and also replace y by $2^t y$ to get that for every positive integer s and t,

$$\begin{aligned}\left\|4^{-t}F_e(2^t y) - 4^{-(s+t)}F_e(2^{s+t}y)\right\| &= 4^{-t}\left\|F_e(2^t y) - 4^{-s}F_e(2^{s+t}y)\right\| \\ &\leq 4^{-(t+1)}\sum_{i=0}^{s-1} 4^{-i}\phi(2^{i+t}y, 2^{i+t}y) \\ &\leq \frac{1}{4}\sum_{i=0}^{\infty} 4^{-(i+t)}\phi(2^{i+t}y, 2^{i+t}y).\end{aligned}$$

According to the condition (3.5), the convergence of $\sum_{i=0}^{\infty} 2^{-i}\phi(2^i y, 2^i y)$ implies that $\sum_{i=0}^{\infty} 4^{-(i+t)}\phi(2^{i+t}y, 2^{i+t}y)$ approaches zero as $s \to \infty$. Therefore, $\{4^{-s}F_e(2^s y)\}_{s=1}^{\infty}$ is a Cauchy sequence in a Banach space. We may define a function $T_e : X \to Y$ as

$$T_e(y) = \lim_{s\to\infty} 4^{-s}F_e(2^s y) = \lim_{s\to\infty} 4^{-s}f_e(2^s y)$$

for all $y \in X$. By taking $s \to \infty$ in (3.14), we arrive at the inequality

$$\|F_e(y) - T_e(y)\| \leq \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\phi(2^i y, 2^i y).$$

Moreover, by the definition of F_e, one get that

$$\begin{aligned}\|f_e(y) - T_e(y)\| &\leq \|F_e(y) + f(0) - T_e(y)\| \\ &\leq \|F_e(y) - T_e(y)\| + \|f(0)\| \\ &\leq \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\phi(2^i y, 2^i y) + \|f(0)\|.\end{aligned}$$

In a similar manner, $\{8^{-s}F_o(2^s y)\}_{s=1}^{\infty}$ is proved to be a convergent sequence in the Banach space. Define a function $T_o : X \to Y$ by

$$T_o(y) = \lim_{s\to\infty} 8^{-s}F_o(2^s y) = \lim_{s\to\infty} 8^{-s}f_o(2^s y)$$

for all $y \in X$. Then

$$\|f_o(y) - T_o(y)\| \leq \frac{1}{8}\sum_{i=0}^{\infty} 8^{-i}\phi(2^i y, 2^i y) + \|f(0)\|.$$

Define a function $T : X \to Y$ by

$$T(y) = T_e(y) + T_o(y)$$

for all $y \in X$. Thus it follow from the previous relations that

$$\begin{aligned} \|f(y) - T(y)\| &\leq \|f_e(y) - T_e(y)\| + \|f_o(y) - T_o(y)\| \\ &\leq \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\phi(2^i y, 2^i y) + \frac{1}{8}\sum_{i=0}^{\infty} 8^{-i}\phi(2^i y, 2^i y) \\ &\quad +2\|f(0)\| \end{aligned} \tag{3.15}$$

for all $y \in X$. Next, we will prove that T satisfies (3.1). We define the even part and odd part of Df by $Df_e(x, y) = \frac{1}{2}(Df(x, y) + Df(-x, -y))$ and $Df_o(x, y) = \frac{1}{2}(Df(x, y) - Df(-x, -y))$. For a positive integer s, putting $(x, y) = (2^s x, 2^s y)$ into the above equations to obtain the relations

$$\|Df_e(2^s x, 2^s y)\| \leq \phi(2^s x, 2^s y) \quad \text{and} \quad \|Df_o(2^s x, 2^s y)\| \leq \phi(2^s x, 2^s y).$$

Dividing the above results by 4^s and 8^s, respectively, and taking the limit as $s \to \infty$. We then have $DT_e(x, y) = 0$ and $DT_o(x, y) = 0$ for all $x, y \in X$. Hence, $T = T_e + T_o$ satisfies (3.1). It only remains to show that T is unique. Suppose that there exists another function $T' : X \to Y$ such that T' satisfies (3.1) and (3.8). From Theorem 3.1, we notice that $T_e = A^0 + A^2(x)$ where $A^2(x)$ satisfies the quadratic functional equation and A^0 is a constant, and T_o satisfies the cubic functional equation; therefore, $A^2(rx) = r^2 A^2(x)$ and $T_o(rx) = r^3 T_o(x)$ for every rational number r and for every $x \in X$. Thus,

$$\left\|T(x) - T'(x)\right\| \leq \left\|T_e(x) - T'_e(x)\right\| + \left\|T_o(x) - T'_o(x)\right\|.$$

For any positive integer s and for each $x \in X$,

$$\begin{aligned} \left\|T_e(x) - T'_e(x)\right\| &= \left\|A^0 + A^2(x) - A^0 - A'^2(x)\right\| \\ &= 4^{-s}\left\|A^2(2^s x) - A'^2(2^s x)\right\| \\ &= 4^{-s}\left\|T_e(2^s x) - T'_e(2^s x)\right\| \\ &\leq 4^{-s}\|f_e(2^s x) - T_e(2^s x)\| + 4^{-s}\left\|f_e(2^s x) - T'_e(2^s x)\right\| \\ &\leq \frac{1}{2}\sum_{i=0}^{\infty} 4^{-(i+s)}\phi(2^{i+s} x) + 2 \cdot 4^{-s}\|f(0)\|. \end{aligned}$$

Taking the limit as $s \to \infty$, we have $\|T_e(x) - T'_e(x)\| \leq 0$. Thus $T_e(x) = T'_e(x)$ for all $x \in X$. Similarly, we can show that $T_o(x) = T'_o(x)$ for all

$x \in X$. Hence $T(x) = T'(x)$ for all $x \in X$.

For the case when the condition (3.6) holds, The proof can be stated in a similar manner. We start the proof by substituting (x, y) by $(\frac{-y}{2}, \frac{y}{2})$ in (3.7) and by $f(0) = 0$ to get that

$$\left\| f(y) - 6f\left(\frac{y}{2}\right) + 2f\left(\frac{-y}{2}\right) \right\| \leq \phi(\frac{y}{2}, \frac{y}{2}).$$

Applying the definitions of f_e and f_o to the previous equation. It yields

$$\left\| f_e(y) - 4f_e\left(\frac{y}{2}\right) \right\| \leq \phi(\frac{y}{2}, \frac{y}{2})$$

and

$$\left\| f_o(y) - 8f_e\left(\frac{y}{2}\right) \right\| \leq \phi(\frac{y}{2}, \frac{y}{2}).$$

We extend the two inequalities to

$$\left\| f_e(y) - 4^s f_e(2^{-s}y) \right\| \leq \frac{1}{4} \sum_{i=1}^{s} 4^i \phi(2^{-i}y, 2^{-i}y)$$

and

$$\left\| f_o(y) - 8^s f_o(2^{-s}y) \right\| \leq \frac{1}{8} \sum_{i=1}^{s} 2^i \phi(2^{-i}y, 2^{-i}y).$$

for a positive integer s and for all $y \in X$. The rest of the proof can be produced in a similar fashion. □

Corollary 3.3. *If a function $f : X \to Y$ satisfies*

$$\|Df(x, y)\| \leq \varepsilon, \tag{3.16}$$

for all $x, y \in X$ and for some $\varepsilon > 0$, then there exists a unique function $T : X \to Y$ that satisfies (3.1) and

$$\|f(y) - T(y)\| \leq \frac{10}{27}\varepsilon + 2\|f(0)\| \tag{3.17}$$

for all $y \in X$.

Proof. We can follow the proof as the Theorem 3.2 by letting $\phi(x, y) = \varepsilon$ for all $x, y \in X$. According to the condition (3.5), it follows from the theorem that there exists a unique function $T; X \to Y$ such that

$$\begin{aligned} \|f(y) - T(y)\| &\leq \|f_e(y) - T_e(y)\| + \|f_o(y) - T_o(y)\| \\ &\leq \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i}\varepsilon + \frac{1}{8}\sum_{i=0}^{\infty} 8^{-i}\varepsilon + 2\|f(0)\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{7} + 2\|f(0)\| \\ &= \frac{10}{27}\varepsilon + 2\|f(0)\| \end{aligned}$$

for all $y \in X$. □

Corollary 3.4. *If a function $f : X \to Y$ satisfies*

$$\|Df(x, y)\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{3.18}$$

for all $x, y \in X$ and for some $\varepsilon > 0$ where p is a positive real number with $p < 1$ or $p > 3$, then there exists a unique function $T : X \to Y$ that satisfies (3.1) and

$$\|f(y) - T(y)\| \leq \frac{4\varepsilon\,|6 - 2^p|}{|4 - 2^p|\,|8 - 2^p|}\,\|y\|^p + 2\|f(0)\| \tag{3.19}$$

for all $y \in X$.

Proof. From the Theorem 3.2, let $\phi(x, y) = \varepsilon\left(\|x\|^p + \|y\|^p\right)$ for all $x, y \in X$. If $p < 1$, then the condition (3.5) holds. Applying the theorem 3.2, we then get

$$\begin{aligned} \|f(y) - T(y)\| &\leq \frac{1}{4}\sum_{i=0}^{\infty} 4^{-i} \cdot 2\varepsilon\left\|2^i y\right\|^p + \frac{1}{8}\sum_{i=0}^{\infty} 8^{-i} \cdot 2\varepsilon\left\|2^i y\right\|^p + 2\|f(0)\| \\ &= \frac{2\varepsilon\|y\|^p}{4 - 2^p} + \frac{2\varepsilon\|y\|^p}{8 - 2^p} + 2\|f(0)\| \\ &= \frac{4\varepsilon(6 - 2^p)}{(4 - 2^p)(8 - 2^p)}\|y\|^p + 2\|f(0)\| \end{aligned}$$

for all $y \in X$. It can be checked that for $p > 3$, the condition (3.6) holds. We therefore obtain a similar result. □

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